Expectation, Variance, and Covariance

Jake Anderson

Outline

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So when you see that college graduates earn more, is that because of the education itself, or because of something else correlated with education?

$\Longrightarrow$ To sort this out, we need three tools: **expectation** (what is the average wage?), **variance** (how much do wages vary?), and **covariance** (how do education and experience move together?).

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$\implies \mu$ is a *fixed number* describing the distribution. Do not confuse it with the sample mean $\bar{x}$, which varies from sample to sample.

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$\Longrightarrow$ The expected value need not be a value X can actually take. It is the long-run average, not a prediction for any single draw.

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Rules 1+2 combined: For constants a, b, c:

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

Linearity in Action: Hourly Wages

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$\implies$ Constants scale and shift the expected value. No surprises here. The interesting part starts with *spread*.

Wages Callback: Expected Wages by Education

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$$\mathbb{E}[W \mid Educ = 12] = \$18, \qquad \mathbb{E}[W \mid Educ = 16] = \$30$$

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$\implies$ The raw wage gap reflects *both* education and experience differences. To separate these effects, we will need covariance. But first, we need a way to measure **spread**.

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The gap between $\mathbb{E}[XY]$ and $\mathbb{E}[X]\mathbb{E}[Y]$ has a name. We will get to it shortly.

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Interpretation: Larger variance $\implies$ the values of X are more spread out around μ. Variance is always ≥ 0, and equals zero only when X is constant.

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$\implies \sigma_X = \sqrt{1} = 1$. On average, values of X deviate from the mean by about 1 unit.

Variance Under Linear Transformation

The rule you will use constantly:

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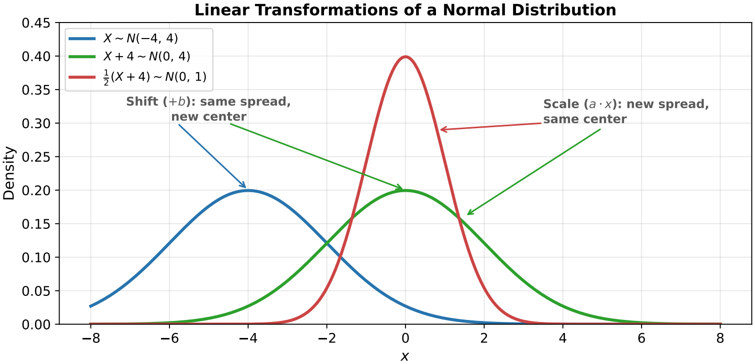
Intuition: If you double every value ($a = 2$), deviations from the mean also double, so squared deviations quadruple.

Scaling vs. Shifting: Visualized

Start with $X \sim N(-4, 4)$. First shift, then scale:



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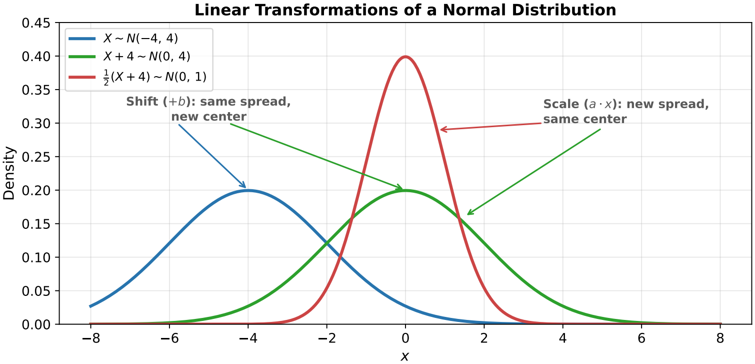

- **Shift** $(X + 4)$: moves the center to $\mu = 0$, but the spread stays at $\sigma^2 = 4$
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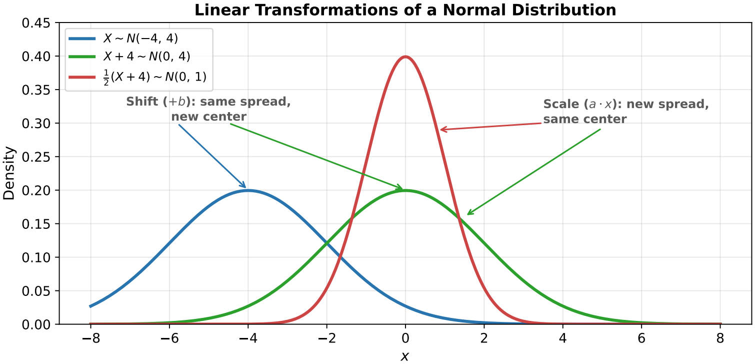

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$\Longrightarrow$ Adding b changes location, not scale. Multiplying by a changes scale.

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Squaring and taking expectations:

$$\text{Var}(Y) = \mathbb{E}[(Y - \mu_Y)^2] = \mathbb{E}[a^2(X - \mu_X)^2] = a^2\mathbb{E}[(X - \mu_X)^2] = a^2\,\text{Var}(X)$$

Variance: Wages in Different Units

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$\implies$ The raise shifts everyone's wage up equally, so the spread does not change. Variance is driven by the random part, not by constants.

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$\Longrightarrow$ Variance tells us there is a lot we cannot explain with education alone. Can we do better by also accounting for experience? That depends on how education and experience **covary**.

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Sign interpretation:

- $\text{Cov}(X, Y) > 0$: X and Y tend to move **in the same direction**
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$\Longrightarrow$ The magnitude of $\text{Cov}(X, Y)$ depends on the units of X and Y, so it does not tell you the *strength* of association. For that, we need correlation.

The $\mathbb{E}[XY]$ Formula, Revisited

Now we can name the gap from earlier. The shortcut formula rearranges to:

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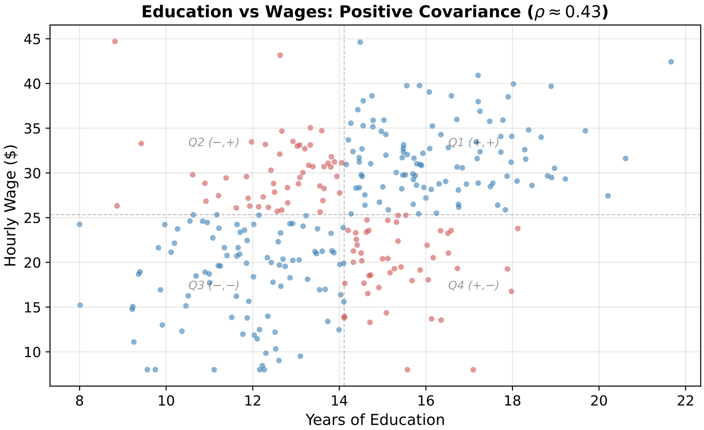
$\implies$ This formula shows why independence simplifies so many proofs: it lets you factor expectations of products.

Covariance: Education and Wages

Education and wages tend to move together. Workers with more education earn more on average.



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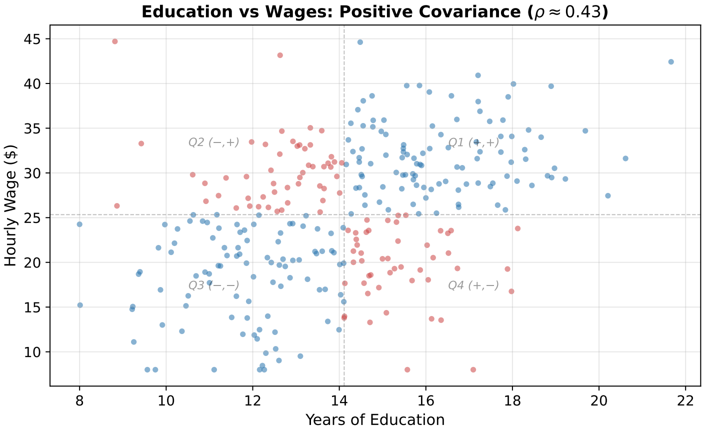
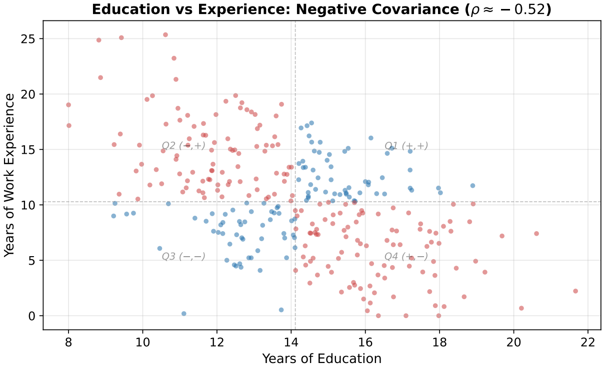


$\text{Cov}(Educ, W) > 0$. Most points cluster in Q1 and Q3 (same-direction deviations from the means).

Covariance: Education and Experience

Here is the key applied insight. At a given age, workers who stayed in school longer have fewer years of work experience.



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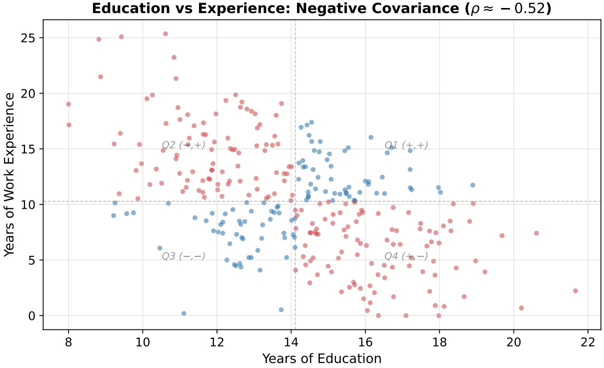

$\text{Cov}(Educ, Exper) < 0$. This negative covariance is exactly what creates problems when we try to estimate the return to education.

Covariance: Properties

1. Symmetry:

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$\implies$ Independence is stronger than zero covariance. We will see this distinction again when we study regression assumptions.

Correlation: Unitless Covariance

Covariance depends on the units of X and Y. To get a **unitless** measure that captures both direction and **strength**, standardize by the standard deviations:

$$\rho_{XY} = \text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \cdot \sigma_Y}$$

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$\Longrightarrow$ This form is useful when you know the correlation and standard deviations.

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$\implies$ Education and experience are both positively associated with wages, but negatively associated with *each other*. This creates a tangle we need to unravel.

Wages Callback: Why Covariance Complicates Things

We know two facts:

- More education $\implies$ higher wages ($\text{Cov}(Educ, W) > 0$)
- More education $\implies$ less experience ($\text{Cov}(Educ, Exper) < 0$)
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So when we compare workers with different education levels, we are also comparing workers with different experience levels. The raw wage gap between a college graduate and a high school graduate reflects *both* differences.

$\implies$ Because education and experience are negatively correlated, we cannot isolate the effect of one without accounting for the other. Later in the course, we will learn how to do this. For now, the takeaway: **when variables are correlated, you cannot study one in isolation**.

Outline

Variance of a Sum: The Formula

For constants a and b:

$$\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) + 2ab\;\mathrm{Cov}(X, Y)$$

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$\Longrightarrow$ The **cross-term** $2ab\;\text{Cov}(X, Y)$ is what makes the variance of a combination depend on the relationship between the variables.

Without the cross-term, you would always predict $\text{Var} = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y)$. This is correct only when $\text{Cov}(X, Y) = 0$.

Variance of a Sum: Derivation

Let $W = aX + bY$, so $\mu_W = a\mu_X + b\mu_Y$:

$$W - \mu_W = a(X - \mu_X) + b(Y - \mu_Y)$$

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$\Longrightarrow$ The cross-term arises from the product of deviations, which is exactly what covariance measures.

Special Cases

Sum ($a = 1, b = 1$):

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$\implies$ "The variance of a sum is the sum of the variances" is only true when $\mathrm{Cov} = 0$. Forgetting the covariance term is one of the most common errors in this course.

Outline

A Simple Wage Model

Suppose wages depend linearly on education and experience:

$$W = \beta_0 + \beta_1 Educ + \beta_2 Exper + \varepsilon$$

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$\implies$ How much wage variation we can explain depends on the variance of each predictor *and* how they covary.

What If You Forgot the Covariance Term?

Naive approach: Ignore the cross-term and compute

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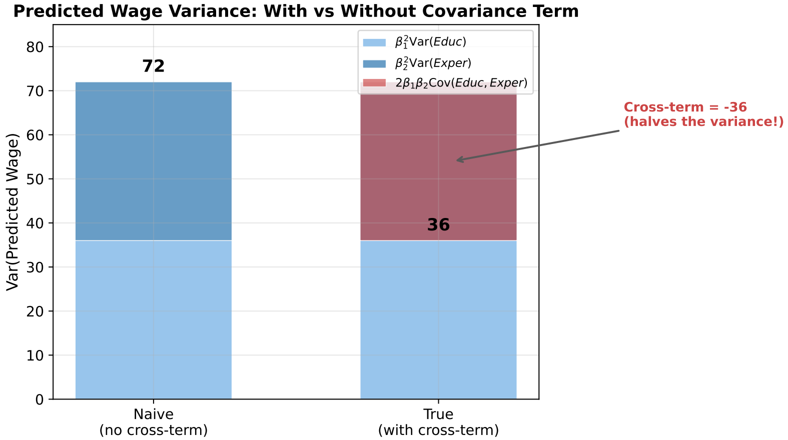
$\implies$ The naive estimate is **double** the true value. Because education and experience move in opposite directions, they partially offset each other, and predicted wages are less variable than you would expect.

Predicted Wage Variance: Visualized



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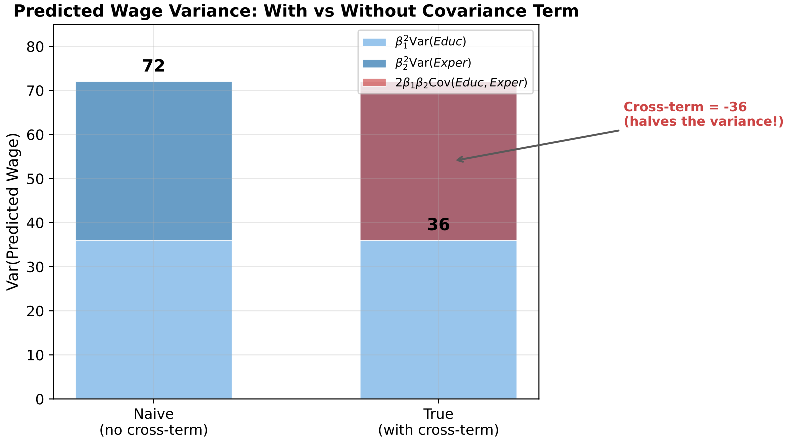

$\Longrightarrow$ The negative covariance between education and experience shrinks the variance of predicted wages. Ignoring the cross-term leads you to overestimate how spread out wages should be.

Outline

Formulas to Internalize

Concept	Formula
Expected value	$\mathbb{E}[X] = \sum_x x\, f(x)$
Linearity of $\mathbb{E}$	$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$
Variance	$\mathrm{Var}(X) = \mathbb{E}[X^2] - [\mathbb{E}[X]]^2$
Variance scaling	$\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$
Covariance	$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
Correlation	$\rho = \mathrm{Cov}(X, Y)/(\sigma_X \sigma_Y)$
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$\Longrightarrow$ These seven formulas are the algebra behind every estimator, standard error, and test statistic in this course. You will use them in every problem set.

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Looking Ahead

These tools connect directly to regression:

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$\Longrightarrow$ Every derivation in Chapters 2 through 16 starts with the rules from today. The education/experience example is an informal version of what we will solve using Ordinary Least Squares (OLS) regression.

Thank you!

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Outline

Derivation: Covariance Shortcut Formula

Start from the definition:

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$$

Expand the product:

$$= \mathbb{E}[XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y]$$

Apply linearity of $\mathbb{E}$ (distribute term by term):

$$= \mathbb{E}[XY] - \mu_Y\,\mathbb{E}[X] - \mu_X\,\mathbb{E}[Y] + \mu_X\mu_Y$$

Substitute $\mathbb{E}[X] = \mu_X$ and $\mathbb{E}[Y] = \mu_Y$:

$$= \mathbb{E}[XY] - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y$$

$$\boxed{\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]}$$