The Normal Distribution, Sampling, and the CLT

Why the Bell Curve Shows Up Everywhere

Jake Anderson

Outline

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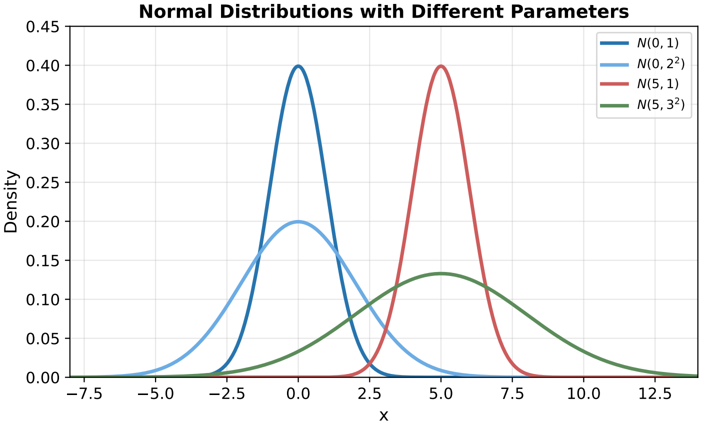
But first: what *is* the normal distribution, and why is it so useful?

What the Normal Distribution Looks Like

The normal distribution is a **symmetric, bell-shaped** curve. It is completely determined by two parameters: μ (center) and σ^2 (spread).



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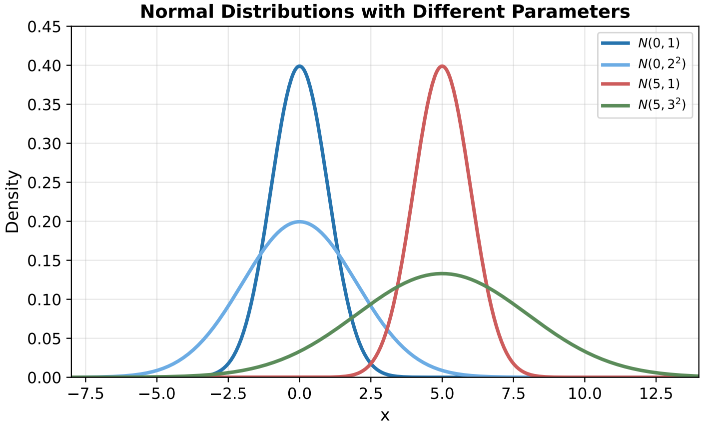

Changing μ shifts the curve left or right. Changing σ makes it wider or narrower. The shape is

The Normal PDF

The function that produces that bell shape is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < x < \infty$$

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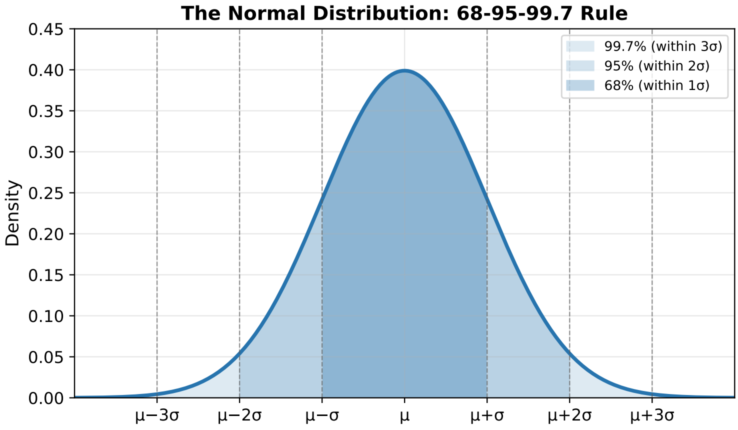
You will never need to evaluate this formula by hand. What you need to know:

- μ determines the **center** (mean = median = mode)
- σ^2 determines the **spread**
- The range is $(-\infty, +\infty)$, but almost all probability is within a few σ of μ

The 68-95-99.7 Rule



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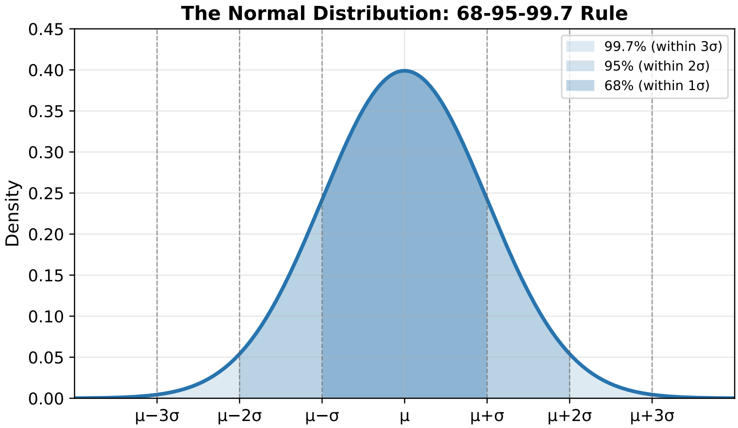

If $X \sim N(\mu, \sigma^2)$:

- About **68%** of values fall within 1 standard deviation: $P(\mu - \sigma \leq X \leq \mu + \sigma) \approx 0.68$
- About **95%** within 2: $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.95$

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The standard normal distribution solves this. If we transform any normal variable so it has mean 0 and variance 1, we only need one table (or one function in software).

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Step 2: Compute its mean and variance using rules you already know:

$$\mathbb{E}[Z] = \mathbb{E}\left[\frac{X - \mu}{\sigma}\right] = \frac{1}{\sigma}\left(\mathbb{E}[X] - \mu\right) = \frac{1}{\sigma}(\mu - \mu) = 0$$

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$\Longrightarrow$ Z is normal with mean 0 and variance 1, so $Z \sim N(0, 1)$.

The Standard Normal CDF: $\Phi(z)$

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For any $X \sim N(\mu, \sigma^2)$, we can compute probabilities by standardizing:

$$P(X \leq a) = \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$P(X > a) = 1 - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$P(a \leq X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

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In practice, you look up Φ in a table (Appendix D, Table 1) or use software:

- R: `pnorm(z)`, Python: `scipy.stats.norm.cdf(z)`

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$\implies$ About 77% of students score between 60 and 85.

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$\implies$ The normal family is "closed" under addition. This will be essential when we study the sampling distribution of $\bar{X}$.

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From Population to Sample

Recall the distinction from last time:

	Population	**Sample**
Quantity	Parameter (fixed, unknown)	Statistic (computed from data)
Mean	$\mu = \mathbb{E}[X]$	$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$
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Because the sample is random, $\bar{X}$ is itself a random variable. If we drew a different sample, we would get a different $\bar{X}$.

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$\Longrightarrow$ Larger samples produce more precise estimates. But what *shape* does the distribution of $\bar{X}$ have?

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This is a linear combination of independent normal random variables (with $a_i = 1/n$ for each). By the “sums of normals” property:

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$\implies$ If the population is normal, the sampling distribution of $\bar{X}$ is **exactly normal** for any sample size.

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That result is the Central Limit Theorem.

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An Experiment with a Skewed Population

Take a population that is **nothing like normal**: the Exponential(1) distribution.

It has mean $= 1$, variance $= 1$, and is heavily right-skewed (probability is concentrated near zero, with a long right tail).

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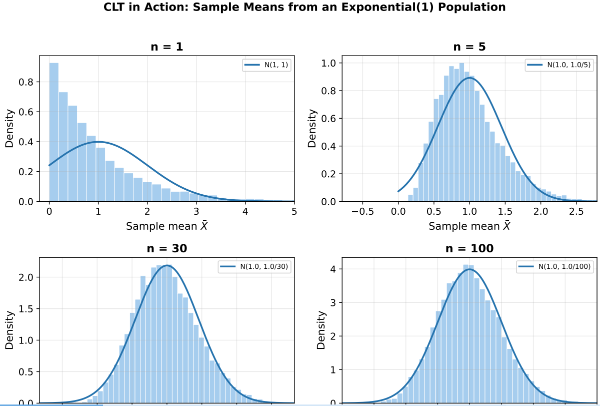
For each sample size n, draw 10,000 samples and plot the histogram of $\bar{X}$:



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The population was nowhere near normal. Yet the distribution of $\bar{X}$ converged to a bell curve as n grew. Why?

The Central Limit Theorem

Central Limit Theorem (CLT)

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with mean μ and finite variance σ^2. Then as $n \to \infty$:

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Equivalently:

$$\bar{X} \overset{\text{approx.}}{\sim} N\left(\mu, \frac{\sigma^2}{n}\right) \quad \text{for large } n$$

The population can be skewed, bimodal, discrete, continuous, bounded, unbounded. It does not need to be normal. The CLT applies as long as the population has **finite variance**.

Example: Applying the CLT

A factory fills cereal boxes. The fill weight per box has $\mu = 368$ g and $\sigma = 15$ g. The distribution of individual box weights is right-skewed (not normal).

A quality inspector weighs $n = 36$ boxes. What is the probability that $\bar{X} > 375$ g?

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Standardize:

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Standardize:

$$Z = \frac{375 - 368}{\sqrt{6.25}} = \frac{7}{2.5} = 2.8$$

$$P(\bar{X} > 375) = 1 - \Phi(2.8) = 1 - 0.9974 = \boxed{0.0026}$$

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A factory fills cereal boxes. The fill weight per box has $\mu = 368$ g and $\sigma = 15$ g. The distribution of individual box weights is right-skewed (not normal).

A quality inspector weighs $n = 36$ boxes. What is the probability that $\bar{X} > 375$ g?

By the CLT: $\bar{X} \overset{\text{approx.}}{\sim} N\left(368,\ \frac{15^2}{36}\right) = N(368,\ 6.25)$

Standardize:

$$Z = \frac{375 - 368}{\sqrt{6.25}} = \frac{7}{2.5} = 2.8$$

$$P(\bar{X} > 375) = 1 - \Phi(2.8) = 1 - 0.9974 = \boxed{0.0026}$$

$\implies$ Even though individual box weights are skewed, we can still compute probabilities about $\bar{X}$ using the normal distribution.

How Large Is “Large Enough”?

The CLT is an asymptotic result ($n \to \infty$), but in practice:

- For **symmetric** populations: $n \geq 5$ often suffices
- For **moderately skewed** populations: $n \geq 30$ is a common rule of thumb
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In econometrics, sample sizes are typically in the hundreds or thousands. $\implies$ The CLT approximation is usually excellent.

Why the CLT Is Foundational for Econometrics

Return to our opening example. Household income is right-skewed, not normal. So how can we do inference about average income, or about regression coefficients?

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$\implies$ Even though household income is skewed, the OLS estimate of the return to education is approximately normal in large samples, because $\hat{\beta}$ is a kind of average, and the CLT applies to averages.

Outline

From the CLT to Hypothesis Testing

The CLT says $\bar{X}$ is approximately normal in large samples. That lets us write:

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Three distributions built from normals handle this and related situations. They are tools we will unpack fully when we reach hypothesis testing and confidence intervals (Topics 9–10). For now, just see how they are constructed from normal random variables.

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Student's t. If $W \sim \chi^2(m)$ is independent of Z: $t = Z/\sqrt{W/m} \sim t(m)$

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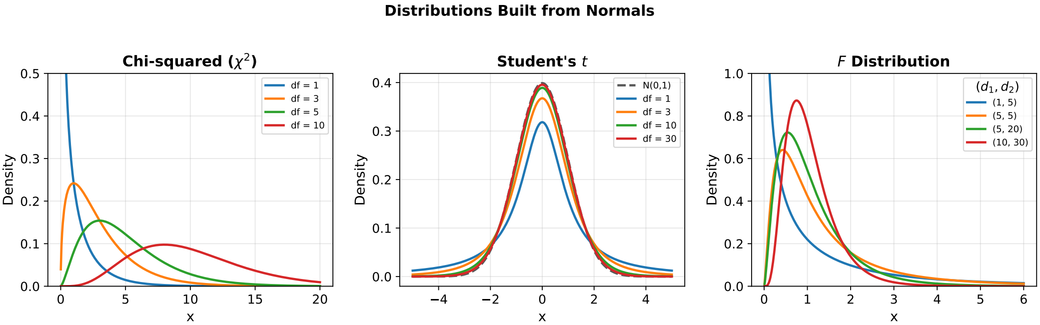
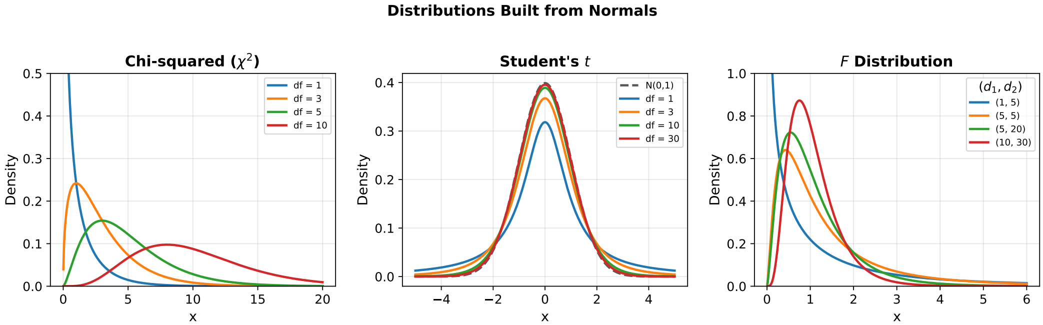
F distribution. If $W_1 \sim \chi^2(m_1)$, $W_2 \sim \chi^2(m_2)$ are independent: $F = \frac{W_1/m_1}{W_2/m_2} \sim F(m_1, m_2)$

- Always positive, right-skewed
- Connection: if $t \sim t(m)$, then $t^2 \sim F(1, m)$

Visualizing the Three Distributions



Visualizing the Three Distributions



The χ^2 and F are always positive and right-skewed. The t is symmetric, with tails that approach the normal as df increases.

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$\implies$ The CLT is the reason we can do inference in econometrics without knowing the true error distribution.

Thank you!

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