Properties of OLS Estimators and the Gauss-Markov Theorem

Why Should We Trust These Estimates?

Jake Anderson

Econ 103, Lecture 4

Outline

Where We Stand

Last time, we estimated the food expenditure model ($N = 40$ Australian households):

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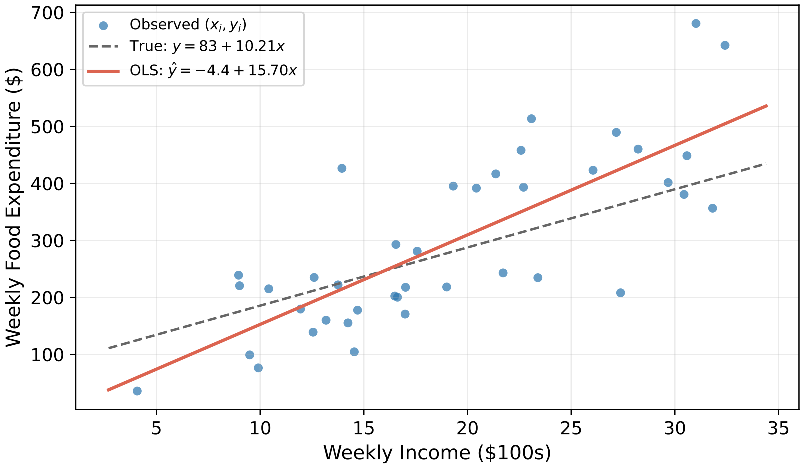
$\implies$ b_2 is a **random variable**. It has a distribution, a mean, and a variance.

One Sample, One Estimate



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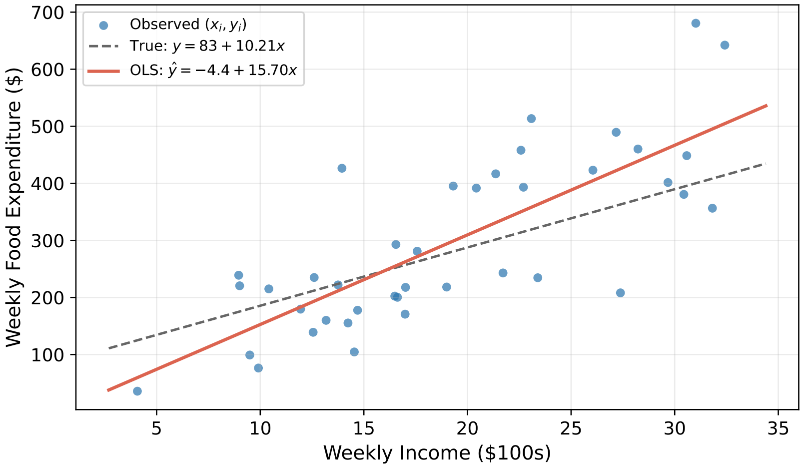
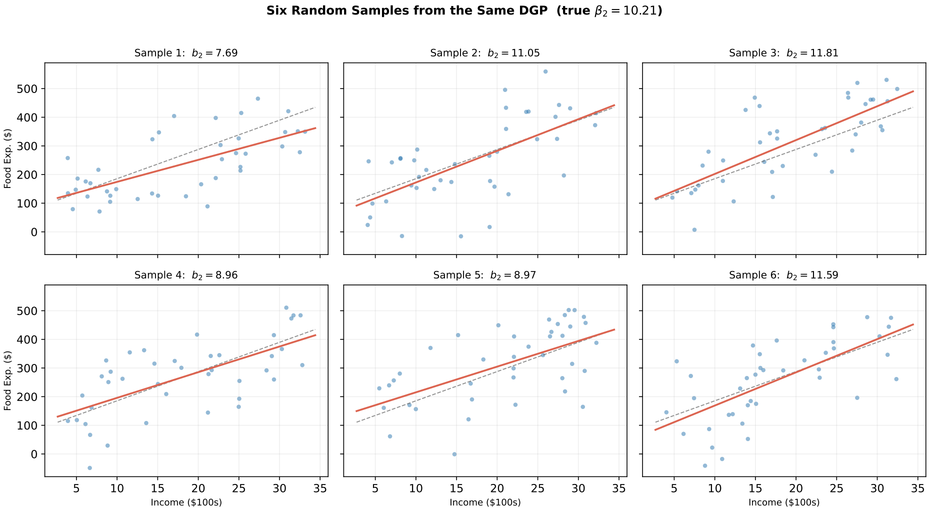


The gray dashed line is the truth. The red line is our estimate from one sample. They are close but not the same.

Different Samples, Different Estimates



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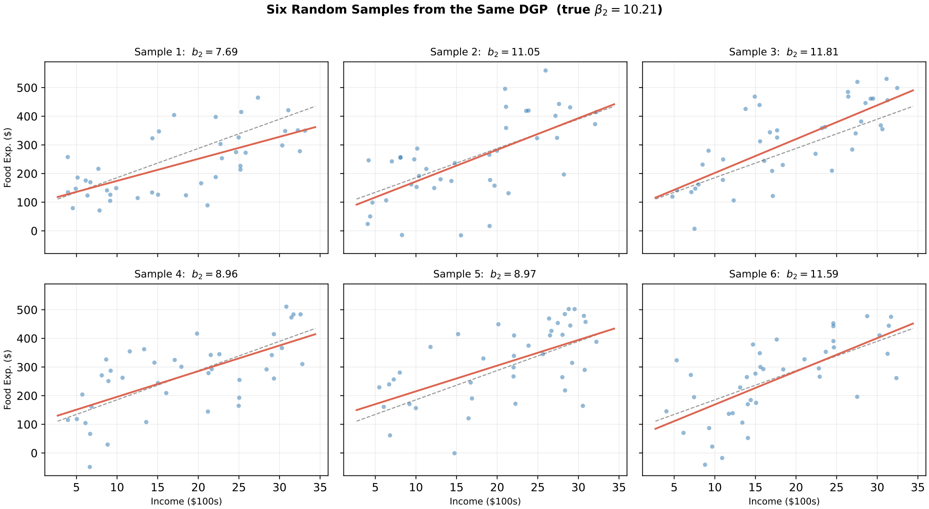
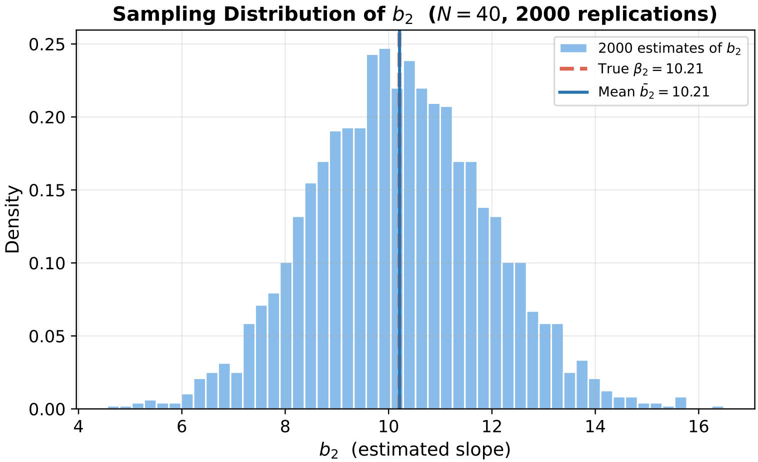
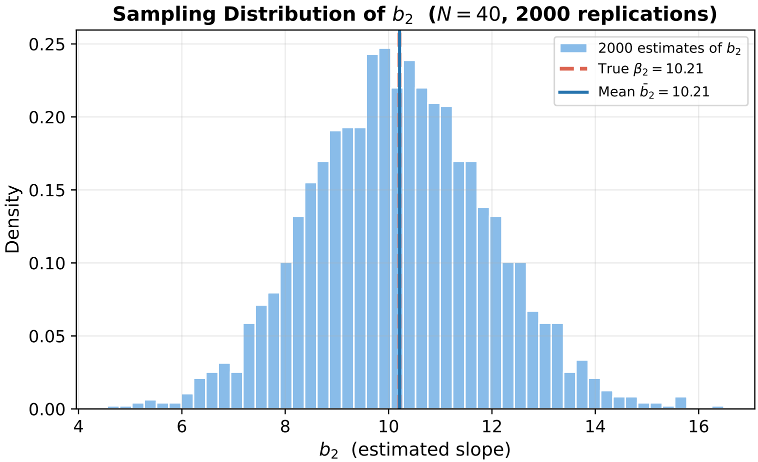


Suppose the true slope is $\beta_2 = 10.21$. Same DGP, same β_2, but each sample gives a different b_2. This

The Sampling Distribution





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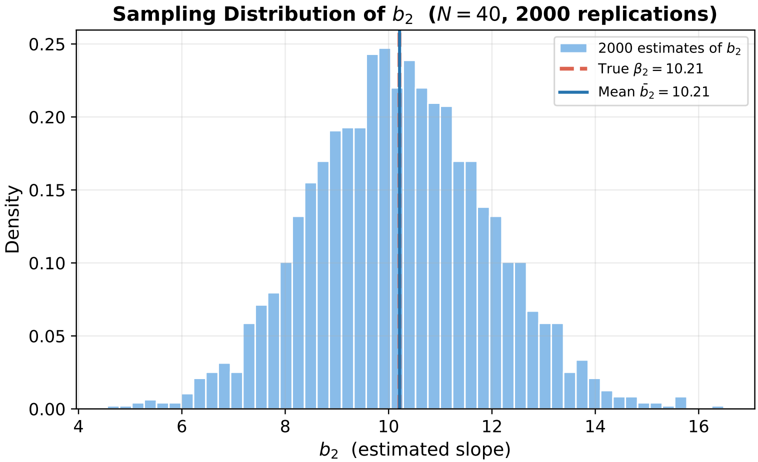

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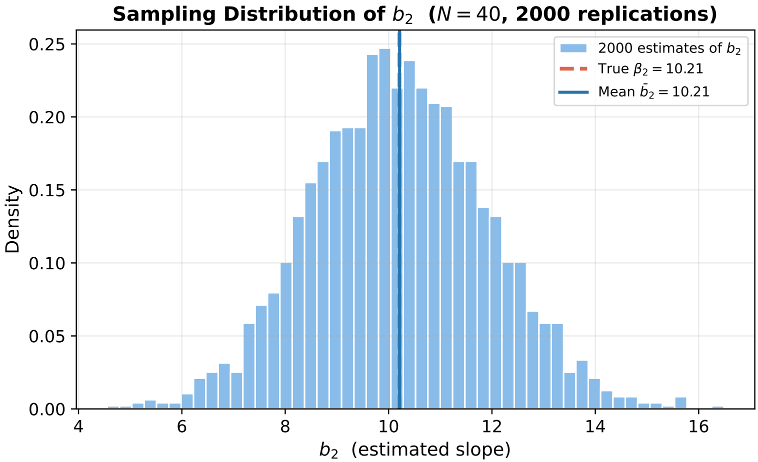

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Rewriting b_2 as a Weighted Sum of the y_i's

To prove b_2 is unbiased, we first need to rewrite it in a form where we can take expectations. Start from the formula you already know:

$$b_2 = \frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

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$\implies$ b_2 is a **linear estimator**: it is a linear function of the data $y_1, y_2, \ldots, y_N$.

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We will use both of these in the unbiasedness proof.

Decomposing b_2: Signal + Noise

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$$= \beta_1 \underbrace{\sum w_i}_{=0} + \beta_2 \underbrace{\sum w_i x_i}_{=1} + \sum w_i e_i$$

$$= \beta_2 + \sum_{i=1}^{N} w_i e_i$$

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$\Longrightarrow$ The simulation histogram was centered on $\beta_2 = 10.21$. That is exactly what unbiasedness predicts.

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By **SR2**: $\text{E}(e_i) = 0$ for every i.

$$\text{E}(b_2) = \beta_2 + \sum_{i=1}^{N} w_i \cdot 0$$

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$\implies$ OLS is **unbiased** for β_2. On average, across all possible samples, b_2 hits the true slope. The same argument shows $\text{E}(b_1) = \beta_1$.

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$\Longrightarrow$ Unbiasedness is not automatic. It depends on the assumptions being correct.

Why Variance Tells Us About Precision

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$\implies$ We want to know: what determines $\text{Var}(b_2)$?

Deriving Var(b_2)

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By **SR3** (homoskedasticity): Var(e_i) $= \sigma^2$ for all i.

Substitute $w_i = \dfrac{x_i - \bar{x}}{\sum(x_j - \bar{x})^2}$:

$$\text{Var}(b_2) = \sigma^2 \sum_{i=1}^{N} w_i^2 = \sigma^2 \sum_{i=1}^{N} \frac{(x_i - \bar{x})^2}{\left[\sum(x_j - \bar{x})^2\right]^2}$$

The numerator sums to $\sum(x_i - \bar{x})^2$, canceling one factor from the denominator:

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Factor 2: Spread of x**-values** (denominator)

- $\sum(x_i - \bar{x})^2$ measures total variation in the explanatory variable
- More spread in x $\implies$ larger denominator $\implies$ smaller $\text{Var}(b_2)$
- Intuition: a line is easier to estimate when the x-values span a wide range

Three Factors That Control Precision

$$\text{Var}(b_2) = \frac{\sigma^2}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

Factor 1: Error variance σ^2 (numerator)

- More noise in the DGP $\implies$ harder to detect the signal $\implies$ larger $\text{Var}(b_2)$
- You cannot control σ^2 (it is a feature of the population)

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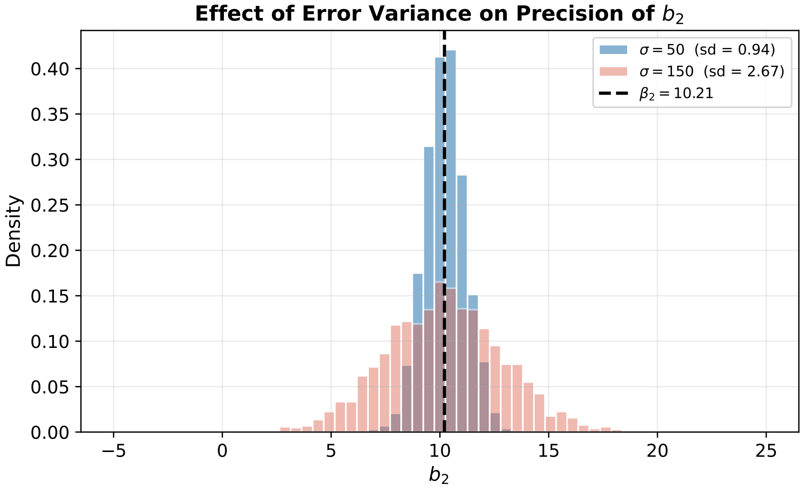
Factor 3: Sample size N

- More observations $\implies$ more terms in $\sum(x_i - \bar{x})^2$ $\implies$ denominator grows
- $\implies$ more data reduces $\text{Var}(b_2)$

Visualizing Factor 1: Error Variance σ^2



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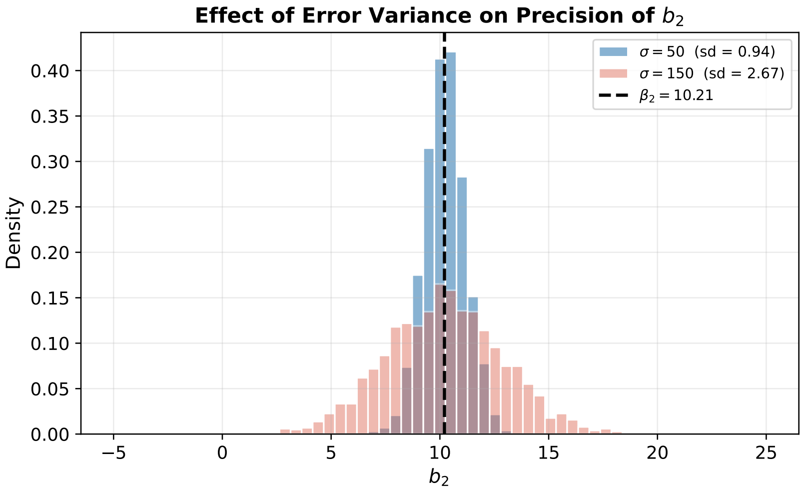
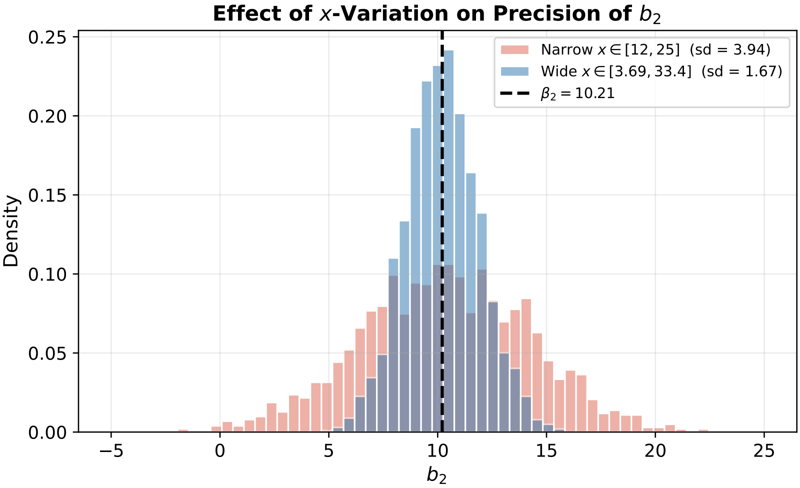


Same N, same x-range. Tripling σ triples the standard deviation of b_2.

Visualizing Factor 2: Spread of x



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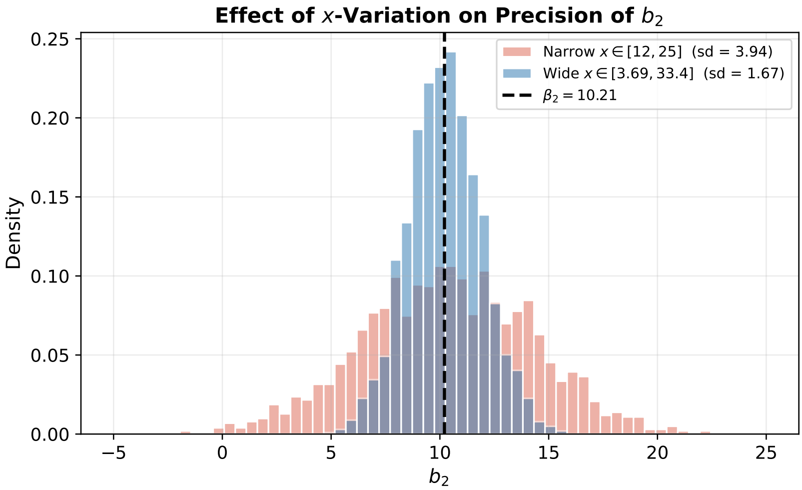
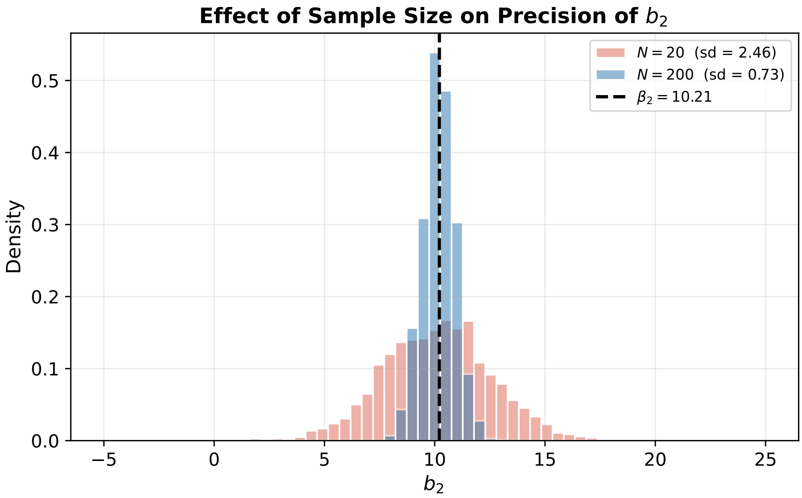


Same N, same σ. Narrowing the x-range concentrates x near its mean, making $\sum(x_i - \bar{x})^2$ smaller

Visualizing Factor 3: Sample Size N



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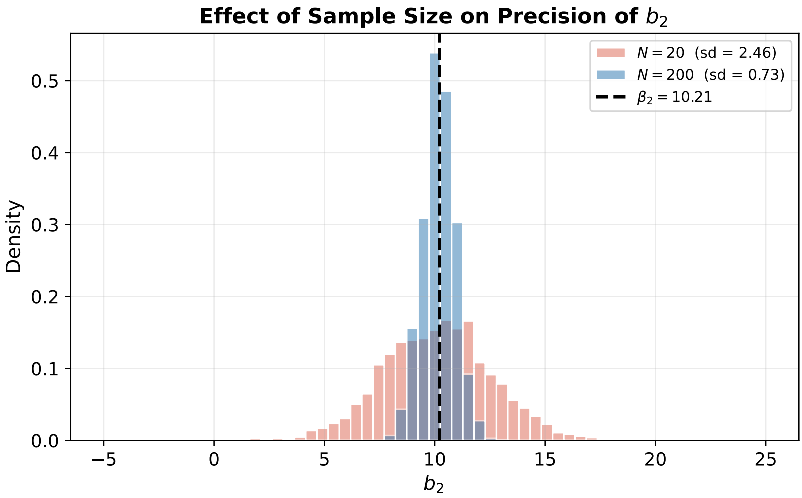

Same σ, same x-range. Going from $N = 20$ to $N = 200$ shrinks the spread dramatically.

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$\Longrightarrow$ The most reliable way to improve precision is to collect more data. If you can design the study, spreading the x-values over a wide range also helps.

Setting Up the Question

We have shown:

- b_2 is a **linear** estimator (weighted sum of y_i's)
- b_2 is **unbiased**: $\mathrm{E}(b_2) = \beta_2$
- b_2 has variance $\mathrm{Var}(b_2) = \sigma^2 / \sum(x_i - \bar{x})^2$

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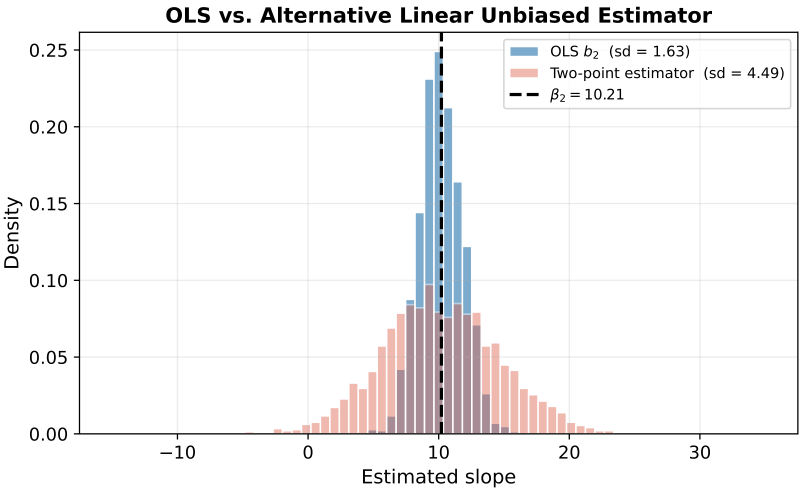
But OLS is not the only linear unbiased estimator. For example, you could use just the two observations with the smallest and largest x-values to draw a line through the data.

Question: Among all possible linear unbiased estimators of β_2, is there one with a *smaller* variance than OLS?

A Concrete Competitor: The Two-Point Estimator



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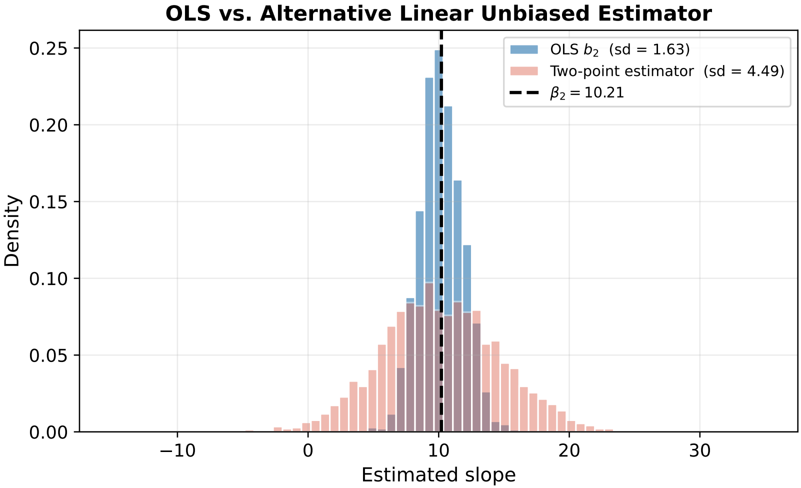

The two-point estimator uses only the observations with the smallest and largest x-values to

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$\Longrightarrow$ You cannot do better than OLS without either (a) giving up linearity, (b) accepting bias, or (c) violating one of SR1–SR4.

Proof Sketch: Setup

Consider any other linear unbiased estimator $\tilde{b}_2 = \sum c_i y_i$ where the c_i are constants (possibly different from the OLS weights w_i).

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$\implies$ If $d_i = 0$ for all i, we are back to OLS. Any nonzero d_i is a *departure* from OLS.

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$\Longrightarrow$ Any departure from OLS weights adds variance. OLS is the minimum.

What Gauss-Markov Does and Does Not Say

What it says:

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- OLS works well when the assumptions are violated
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$\Longrightarrow$ Gauss-Markov gives you a strong reason to use OLS *when the assumptions hold*.

Putting It All Together

Property	Result	Requires
Linearity	$b_2 = \sum w_i y_i$	Definition of OLS
Decomposition	$b_2 = \beta_2 + \sum w_i e_i$	SR1
Unbiasedness	$\text{E}(b_2) = \beta_2$	SR1, SR2
Variance formula	$\text{Var}(b_2) = \dfrac{\sigma^2}{\sum(x_i - \bar{x})^2}$	SR1–SR4
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Next time: how to *use* these results. We will estimate σ^2, compute standard errors, and build confidence intervals for β_2.

Thank you!

jakeanderson@g.ucla.edu