# Interval Estimation and Confidence Intervals

## From Point Estimates to Ranges of Plausible Values

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# Outline


1. The Precision Problem
2. From Normal to $t$: Building the Interval
3. Interpretation: What Does "95% Confidence" Mean?
4. Example: Food Expenditure
5. Confidence Intervals for Linear Combinations
6. Summary

# Where We Left Off

Last time we estimated the food expenditure model and reported:

$$\widehat{\text{food}} = \underset{(43.41)}{83.42} + \underset{(2.09)}{10.21}\,\text{income}$$

We know $b_2 = 10.21$ with $\text{se}(b_2) = 2.09$.

But what does that *tell* us? How confident are we in 10.21?

- Could the true return to income be \$5 per \$100? *(5.21 / 2.09 = 2.49 se's away)*
- Could it be \$15? *(4.79 / 2.09 = 2.29 se's away)*
- Could it be \$0? *(10.21 / 2.09 = 4.89 se's away)*

$\Longrightarrow$ A point estimate and a standard error are the ingredients. A **confidence interval** is the recipe that turns them into a range of plausible values for $\beta_2$.

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# Step 1: $b_2$ Is Normally Distributed

Under assumptions SR1–SR6 (including normality of errors), we showed:

$$b_2 \sim N\left(\beta_2,\ \frac{\sigma^2}{\sum(x_i - \bar{x})^2}\right)$$

Standardize by subtracting the mean and dividing by the standard deviation:

$$Z = \frac{b_2 - \beta_2}{\sqrt{\sigma^2 / \sum(x_i - \bar{x})^2}} \sim N(0, 1)$$

If we knew $\sigma^2$, we could write:

$$P(-1.96 \leq Z \leq 1.96) = 0.95$$

and rearrange to get an interval for $\beta_2$. But we do not know $\sigma^2$.

# Step 2: Replace $\sigma^2$ with $\hat{\sigma}^2$

We **estimated** $\sigma^2$ in Topic 8 using the residuals:

$$\hat{\sigma}^2 = \frac{\sum \hat{e}_i^2}{N-2}$$

Plugging this into the denominator changes $Z$ into a new statistic:

$$t = \frac{b_2 - \beta_2}{\underbrace{\sqrt{\hat{\sigma}^2 / \sum (x_i - \bar{x})^2}}_{\text{se}(b_2)}} = \frac{b_2 - \beta_2}{\text{se}(b_2)}$$

Replacing the known $\sigma$ with $\hat{\sigma}$ in the denominator adds estimation uncertainty. The resulting statistic follows a $t$**-distribution** rather than the standard normal.

# Step 3: The $t$-Distribution

$$t = \frac{b_2 - \beta_2}{\text{se}(b_2)} \sim t_{(N-2)}$$

Where does $N - 2$ come from?

- We used $N$ residuals to estimate $\hat{\sigma}^2$
- But we spent 2 degrees of freedom estimating $b_1$ and $b_2$
- $\implies N - 2$ degrees of freedom remain

The $t$-distribution looks like the standard normal, but with **heavier tails**:

- Extra tail weight reflects our **uncertainty about** $\sigma$
- As $N \to \infty$, $\hat{\sigma}^2 \to \sigma^2$ and $t_{(N-2)} \to N(0, 1)$

$\implies$ With small samples, we need wider intervals to compensate for estimating $\sigma$.

# What If You Ignore the Extra Uncertainty?

Suppose you use the normal critical value $z_{0.025} = 1.96$ instead of $t_c = 2.024$:

| | **Using** $z = 1.96$ | **Using** $t_c = 2.024$ |
|---|---|---|
| Margin | $1.96 \times 2.09 = 4.10$ | $2.024 \times 2.09 = 4.23$ |
| Interval | $[6.11,\ 14.31]$ | $[5.98,\ 14.44]$ |

The normal-based interval is **too narrow**. It acts as if $\sigma$ were known exactly, ignoring the additional uncertainty from estimating it.

$\Longrightarrow$ In small samples, this interval would fail to capture $\beta_2$ more than 5% of the time. The $t$-distribution corrects for this.

# Step 4: Rearranging for the Interval

Start with the probability statement:

$$P\left(-t_c \ \leq \ \frac{b_2 - \beta_2}{\text{se}(b_2)} \ \leq \ t_c\right) = 1 - \alpha$$

where $t_c = t_{(1-\alpha/2,\ N-2)}$ is the critical value from the $t$-table.

Multiply through by $\text{se}(b_2)$:

$$P\Big( - t_c \cdot \text{se}(b_2) \ \leq \ b_2 - \beta_2 \ \leq \ t_c \cdot \text{se}(b_2)\Big) = 1 - \alpha$$

Rearrange (subtract $b_2$, multiply by $-1$, flip inequalities):

$$\boxed{P\Big[b_2 - t_c \cdot \text{se}(b_2) \ \leq \ \beta_2 \ \leq \ b_2 + t_c \cdot \text{se}(b_2)\Big] = 1 - \alpha}$$

This is the $100(1-\alpha)\%$ **confidence interval** for $\beta_2$.

# The Confidence Interval Formula

$$\text{CI for } \beta_k : \qquad b_k \pm t_c \cdot \text{se}(b_k)$$

where:

- $b_k$ = OLS point estimate (center of the interval)
- $\text{se}(b_k)$ = standard error (measures precision)
- $t_c = t_{(1-\alpha/2,\ N-2)}$ = critical value from the $t$-distribution
- $\alpha$ = significance level (e.g., $\alpha = 0.05$ for a 95% CI)

Three moving parts, and you already know two of them ($b_k$ and $\text{se}(b_k)$). The only new ingredient is $t_c$, which you look up in a table or compute in software.

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# The Repeated Sampling Interpretation

Before we compute our first interval, we need to understand what it will tell us.

“95% confidence” does **not** mean: “there is a 95% probability that $\beta_2$ is in the interval.”

Why not? Because $\beta_2$ is a **fixed, unknown constant**. It is either inside the interval or it is not. There is nothing random about $\beta_2$.

What *is* random? **The interval.**

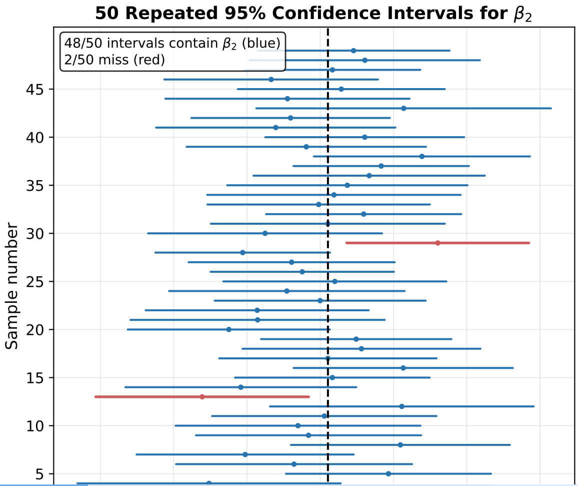
- $b_2$ changes from sample to sample $\implies$ the center moves
- $\text{se}(b_2)$ changes from sample to sample $\implies$ the width changes
- Each sample produces a different interval

The correct interpretation: if we drew many samples and built a 95% CI from each one, **approximately 95% of those intervals would contain the true** $\beta_2$.

$\implies$ Our confidence is in the **procedure**, not in any single interval.

# Visualizing Repeated Sampling

Simulation: draw 50 random samples from the food expenditure DGP, compute a 95% CI for $\beta_2$ from each.



# The Interval Is Random, Not $\beta_2$

## Common misinterpretation

"There is a 95% probability that $\beta_2$ lies in the interval."

## Correct interpretation

"The procedure we used captures the true $\beta_2$ in 95% of repeated samples. We do not know whether this particular interval is one of the 95% or the 5%."

Analogy: a factory produces light bulbs, and 95% work. You buy one. You do not say "there is a 95% probability this bulb works." It either works or it does not. The 95% describes the **factory's track record**, not your particular bulb.

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# 95% CI for the Return to Income

**Data**: $N = 40$ households. Degrees of freedom: $N - 2 = 38$.
**Estimates**: $b_2 = 10.21$, $\text{se}(b_2) = 2.09$.
**Critical value**: For a 95% CI, $\alpha = 0.05$:

$$t_c = t_{(0.975,\ 38)} = 2.024$$

**Interval**:

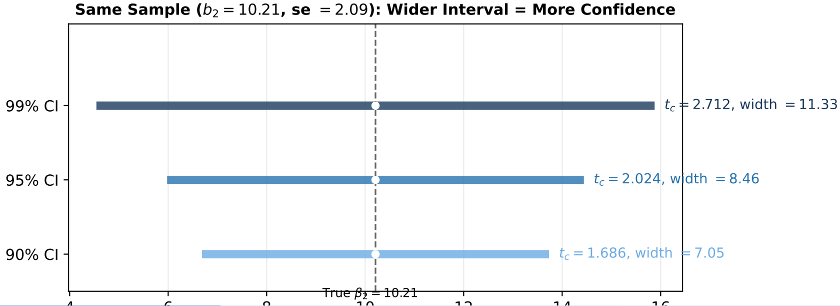
$$\begin{aligned} b_2 \pm t_c \cdot \text{se}(b_2) &= 10.21 \pm 2.024 \times 2.09 \\ &= 10.21 \pm 4.23 \\ &= [5.98,\ 14.44] \end{aligned}$$

We estimate with 95% confidence that an additional \$100 of weekly income increases food expenditure by between \$5.98 and \$14.44.

# Changing the Confidence Level

Same sample, same data. What happens if we change $\alpha$?

| Level | $t_c$ | Interval | Width |
|---|---|---|---|
| 90% ($\alpha = 0.10$) | $t_{(0.95,38)} = 1.686$ | $[6.69,\ 13.73]$ | 7.04 |
| 95% ($\alpha = 0.05$) | $t_{(0.975,38)} = 2.024$ | $[5.98,\ 14.44]$ | 8.46 |
| 99% ($\alpha = 0.01$) | $t_{(0.995,38)} = 2.712$ | $[4.54,\ 15.88]$ | 11.34 |



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# Why Linear Combinations?

We now have a CI for a single parameter. But what if we want a CI for a **prediction** that depends on both $\beta_1$ and $\beta_2$?

**Example**: What is the expected food expenditure for a household earning $2,000/week ($x_0 = 20$)?

$$E(\text{food} \mid x = 20) = \beta_1 + 20\,\beta_2$$

This is a **linear combination**: $\lambda = c_1\beta_1 + c_2\beta_2$ with $c_1 = 1$, $c_2 = 20$.

Point estimate:

$$\hat{\lambda} = b_1 + 20 \cdot b_2 = 83.42 + 20 \times 10.21 = 287.61$$

We predict $287.61 per week. But how precise is this?

# Variance of a Linear Combination

For $\hat{\lambda} = c_1 b_1 + c_2 b_2$, the variance of a sum of correlated random variables is:

$$\text{Var}(\hat{\lambda}) = c_1^2 \ \text{Var}(b_1) + c_2^2 \ \text{Var}(b_2) + 2\, c_1 c_2 \ \text{Cov}(b_1, b_2)$$

Do not forget the **covariance term**. The estimators $b_1$ and $b_2$ are correlated (both come from the same data), so you cannot just add the individual variances.

Where do the numbers come from? Software reports the estimated covariance matrix:

$$\widehat{\text{Cov}}(b_1, b_2) = \begin{pmatrix} \widehat{\text{Var}}(b_1) & \widehat{\text{Cov}}(b_1, b_2) \\ \widehat{\text{Cov}}(b_1, b_2) & \widehat{\text{Var}}(b_2) \end{pmatrix} = \begin{pmatrix} 1884.44 & -85.90 \\ -85.90 & 4.38 \end{pmatrix}$$

For $c_1 = 1$, $c_2 = 20$:

$$\begin{aligned} \widehat{\text{Var}}(\hat{\lambda}) &= (1)^2 \times 1884.44 + (20)^2 \times 4.38 + 2(1)(20)(-85.90) \\ &= 1884.44 + 1752.72 - 3436.13 \\ &= 201.03 \end{aligned}$$

$\text{se}(\hat{\lambda}) = \sqrt{201.03} = 14.18$

# CI for Expected Food Expenditure

The CI formula itself is unchanged: $\hat{\lambda} \pm t_c \cdot \text{se}(\hat{\lambda})$. The only new ingredient is the variance formula for the linear combination.

Same $t_c = 2.024$ (95% CI, $df = 38$):

$$\begin{aligned} 287.61 \pm 2.024 \times 14.18 &= 287.61 \pm 28.69 \\ &= [258.91,\ 316.31] \end{aligned}$$

We estimate with 95% confidence that a household earning $2,000/week spends between $258.91 and $316.31 on food.

This interval reflects our uncertainty about **both** $\beta_1$ and $\beta_2$ simultaneously, plus their covariance.

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# What We Covered

A point estimate without a confidence interval is incomplete. The formula is one line, but the interpretation requires understanding that confidence is about the procedure, not the interval.

**The derivation**: $b_2$ is normal (SR6) $\implies$ standardize $\implies$ replace $\sigma$ with $\hat{\sigma}$ $\implies$ $t$-distribution with $N - 2$ df.

**The formula**: $b_k \pm t_c \cdot \text{se}(b_k)$, where $t_c = t_{(1-\alpha/2,\ N-2)}$.

**The interpretation**: confidence is in the procedure, not in any single interval. Approximately $100(1 - \alpha)\%$ of intervals constructed this way will contain the true parameter.

**The tradeoff**: more confidence $\implies$ wider interval. No free lunch.

**Linear combinations**: for $\lambda = c_1\beta_1 + c_2\beta_2$, use the full variance formula (do not forget the covariance term), then apply the same CI recipe.

**Next**: now that we can build intervals, what if we want a yes-or-no answer? $\implies$ hypothesis testing (Topic 10).

Thank you!

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