F-Tests and Joint Hypothesis Testing

When Individual t-Tests Are Not Enough

Jake Anderson

March 21, 2026

Outline

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Where We Left Off

Last lecture ended with a puzzle. Consider the wage model:

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$\Longrightarrow$ **No.** The two variables are highly correlated, inflating both standard errors. We need a test that evaluates them **together**.

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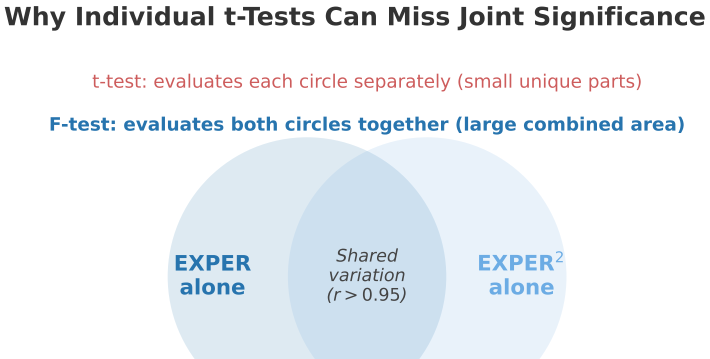

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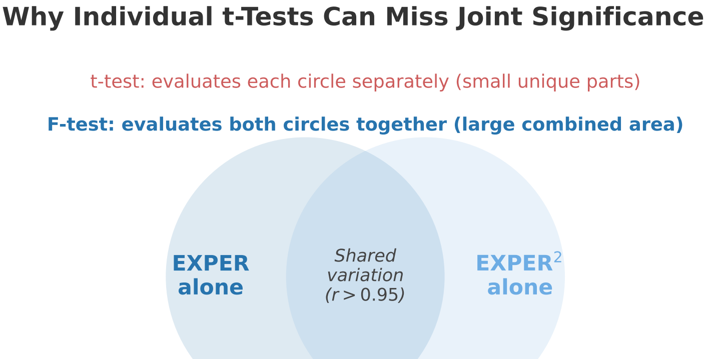

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$\Longrightarrow$ The question is whether $SSE_R - SSE_U$ is large enough to be **statistically meaningful**, or just sampling noise.

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Reject H_0 if $F \geq F_c$, where $F_c = F_{(1-\alpha,\ J,\ N-K)}$.

$\implies$ The numerator measures how much the restrictions **hurt** the fit (per restriction). The denominator is the baseline noise level (per residual degree of freedom).

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$\implies$ The F-test is always **one-sided** (right tail only), because $SSE_R \geq SSE_U$ guarantees $F \geq 0$.

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$J = 2$ restrictions. Degrees of freedom: $N - K = 996$.

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Step 1: Compute F.

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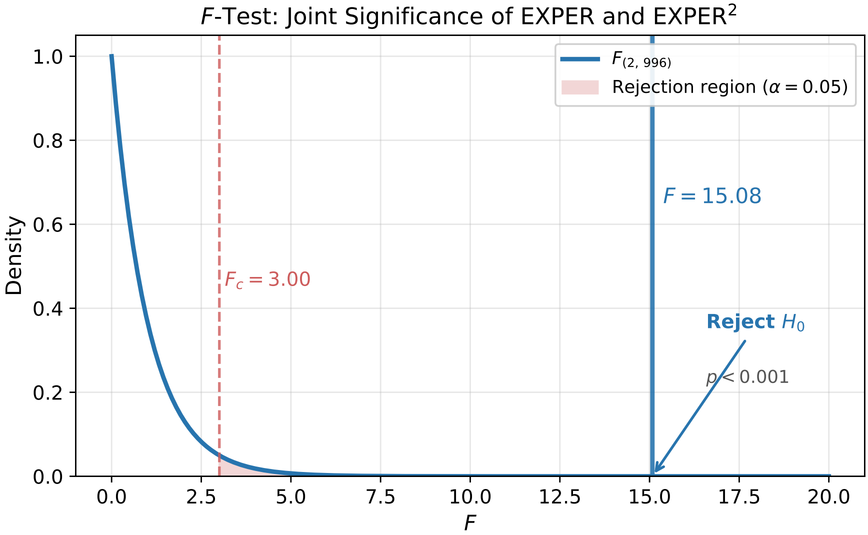
Step 3: Decide.
$15.08 > 3.00 \implies$ **Reject H_0.** The experience terms are jointly significant.

Visualizing the Decision



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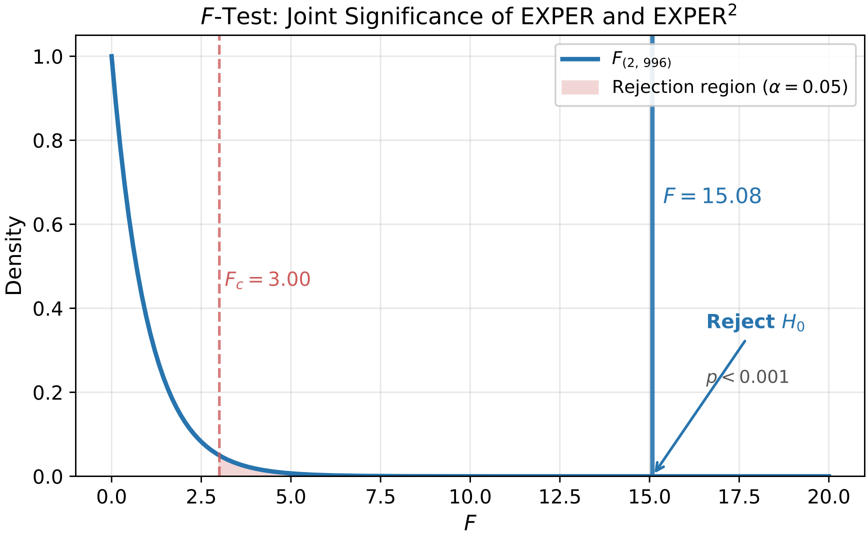

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$\implies$ Always use the unrestricted model's parameter count for K.

What the F-Test Can and Cannot Do

Capability	t**-test**	F**-test**
Single restriction ($J = 1$)	Yes	Yes
Multiple restrictions ($J \geq 2$)	No	Yes
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$\Longrightarrow$ Use t-tests for single restrictions (they handle one-sided alternatives). Use F-tests for joint restrictions ($J \geq 2$).

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$\implies$ This is reported automatically by every regression software package as the “overall F-statistic.”

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$\Longrightarrow$ The overall F-test asks whether the regression mean square is large relative to the residual mean square.

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$\implies$ This is the first thing to check in any regression output. If the overall F-test fails to reject, none of the regressors are jointly significant.

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To test whether $1,900/month is optimal ($\text{ADVERT}_0 = 1.9$):

$$H_0: \beta_3 + 3.8\beta_4 = 1 \qquad H_1: \beta_3 + 3.8\beta_4 \neq 1$$

Beyond "Equals Zero" Restrictions

The F-test handles any linear equality hypothesis, not just "coefficient equals zero."

Example: In the Big Andy's Burger Barn model,

$$\text{SALES} = \beta_1 + \beta_2\,\text{PRICE} + \beta_3\,\text{ADVERT} + \beta_4\,\text{ADVERT}^2 + e$$

The marginal effect of advertising is $\beta_3 + 2\beta_4\,\text{ADVERT}$. Optimal advertising satisfies:

$$\text{marginal revenue} = 1 \implies \beta_3 + 2\beta_4\,\text{ADVERT}_0 = 1$$

To test whether \$1,900/month is optimal ($\text{ADVERT}_0 = 1.9$):

$$H_0: \beta_3 + 3.8\beta_4 = 1 \qquad H_1: \beta_3 + 3.8\beta_4 \neq 1$$

$\implies$ This is a single restriction ($J = 1$), but the restriction is **not** "equals zero." The F-test still works; you just need the SSE_R from the model that imposes $\beta_3 + 3.8\beta_4 = 1$.

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Rearrange to get a standard regression:

$$\underbrace{\text{SALES} - \text{ADVERT}}_{\text{new LHS}} = \beta_1 + \beta_2\,\text{PRICE} + \beta_4\,\underbrace{(\text{ADVERT}^2 - 3.8\,\text{ADVERT})}_{\text{new variable}} + e$$

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$\Longrightarrow$ Run OLS on this transformed equation. Its SSE is SSE_R.

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$0.94 < 3.976 \implies$ **Do not reject H_0.** The data are consistent with \$1,900/month being the optimal advertising level.

Since $J = 1$, this is equivalent to a two-sided t-test: $t = \sqrt{0.94} = 0.97$, $t_c = 1.994$. Same conclusion.

Outline

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$\Longrightarrow$ OLS on this transformed model gives **restricted least squares** estimates.

Properties of Restricted Least Squares

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$\implies$ Use the F-test to check whether the restriction is supported by the data before relying on restricted estimates.

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$\Longrightarrow$ Always verify: $SSE_R \geq SSE_U$. If not, you have the models backward.

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$\Longrightarrow$ Do **not** rely on individual t-tests alone when variables are correlated. The F-test is the correct tool for groups of variables.

Comparison: The Testing Toolkit

Question	J	Test	Statistic
Is $\beta_k = 0$?	1	t-test	$t = \dfrac{b_k}{\text{se}(b_k)}$
Is $\beta_k = c$?	1	t-test	$t = \dfrac{b_k - c}{\text{se}(b_k)}$
Is $c_1\beta_1 + c_2\beta_2 = q$?	1	t-test	$t = \dfrac{\hat{\lambda} - q}{\text{se}(\hat{\lambda})}$
Are $\beta_3 = 0$ and $\beta_4 = 0$?	2	F-test	$F = \dfrac{(SSE_R - SSE_U)/J}{SSE_U/(N-K)}$
Are all slopes $= 0$?	$K-1$	F-test	$F = \dfrac{(SST - SSE)/(K-1)}{SSE/(N-K)}$

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Are $\beta_3 = 0$ and $\beta_4 = 0$?	2	F-test	$F = \dfrac{(SSE_R - SSE_U)/J}{SSE_U/(N-K)}$
Are all slopes $= 0$?	$K-1$	F-test	$F = \dfrac{(SST - SSE)/(K-1)}{SSE/(N-K)}$

$\Longrightarrow$ The t-test is for single restrictions. The F-test is for one or more restrictions. When $J = 1$, $F = t^2$.

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4. **Restricted least squares.**
   Imposing a restriction always reduces variance but introduces bias unless the restriction is correct.
5. **Practical rule:**
   Before dropping a group of correlated variables, run the F-test. Individual t-tests can mislead when regressors are correlated.

Thank you!

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