Introduction to Fixed Effects

Why One Regression Line Isn't Enough

Jake Anderson

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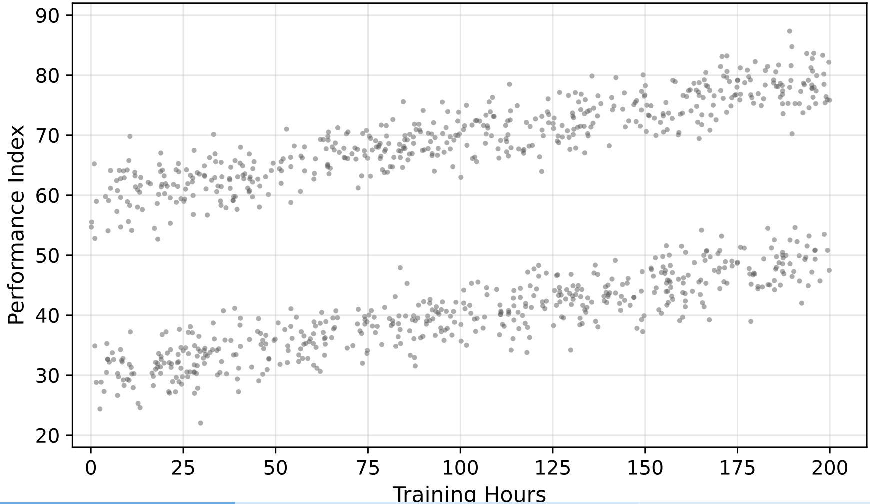
The Data

A coach tracks **training hours** vs. **performance** for their trainees.



The Setup

It turns out there are **two teams**: Varsity and Junior Varsity (JV). Same training program, different baseline ability.
Let's assume the following:

- JV players have a baseline of 30 "skill points"; Varsity have 60
- Each additional 10 hours of training $\rightarrow +1$ performance point (slope $= 0.1$)

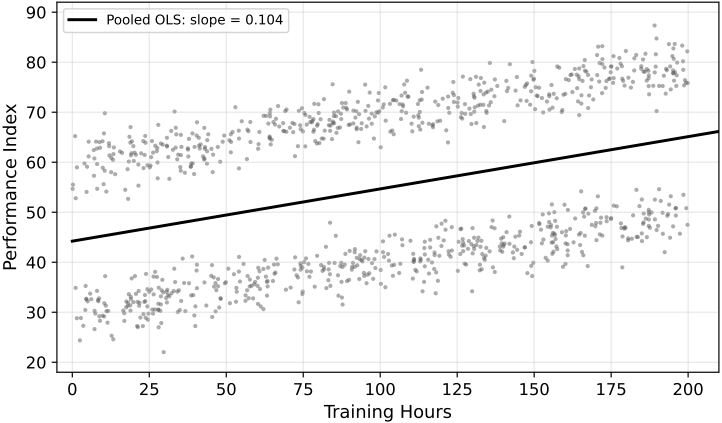
If we ignore team membership and run a single regression:

$$\text{Performance}_i = \underbrace{\beta_0}_{\substack{\text{Assumes the}\\ \text{same intercept}\\ \text{for both teams!}}} + \underbrace{\beta_1}_{\substack{\text{Same slope}\\ \text{for both — OK}}} \text{Hours}_i + \varepsilon_i$$

But JV starts at 30 and Varsity starts at 60 — a single β_0 **cannot be right for both groups**.

Question: What goes wrong when we force one intercept on data with two?

What Happens If We Run Naive OLS?



Pooled OLS: slope ≈ 0.10 — looks correct! But this only works because the sample is balanced (50/50) and x is identically distributed across groups. What if that changes?

First Check: Are Groups Sampled Differently?

In practice, the distribution of x often differs across groups.

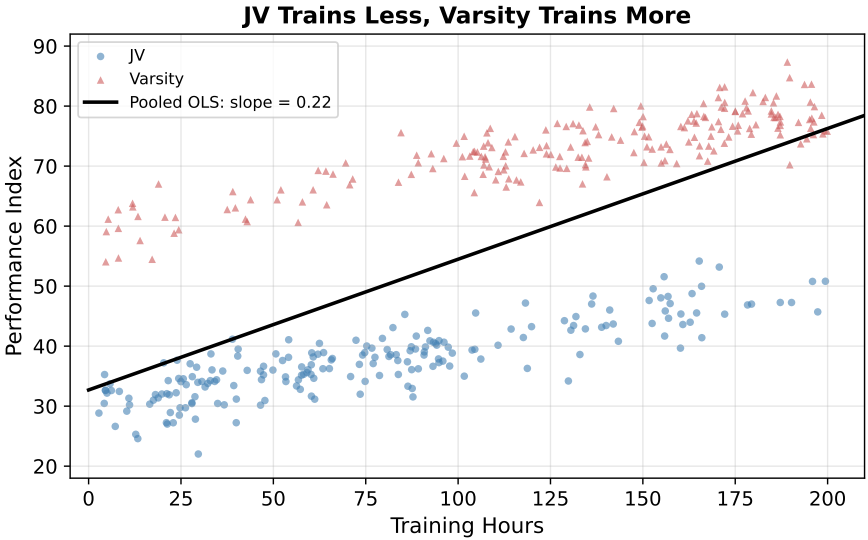
- Varsity players may train **more** (selection into training)
- Or JV players may train more (catching up)

Key question: Does $\text{Cov}(\text{Group}, x) \neq 0$?

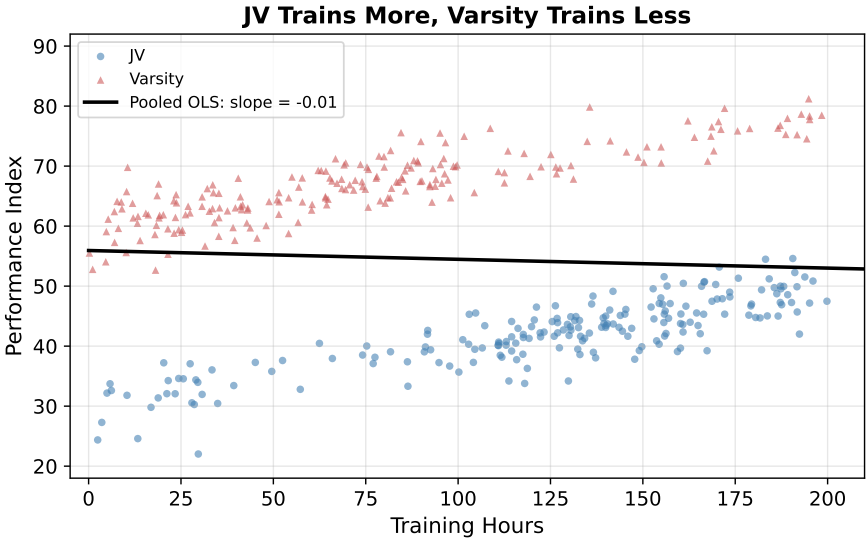
If so, pooled OLS conflates the group effect (α_j) with the treatment effect (β).

This is **omitted variable bias**.

Scenario: Varsity Trains More



Scenario: JV Trains More (Catching Up)



The Omitted Variable Bias

Let's map this to the OVB framework you already know. Let $X_2 =$ Group / team membership (OV)

Short regression (what we run $\rightarrow$ omits X_2):

$$Y_i = \beta_0 + \beta_1 X_{1i} + u_i$$

Long regression (what we should run $\rightarrow$ includes X_2):

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i$$

Auxiliary regression (relationship between included and omitted):

$$X_{2i} = \delta_0 + \delta_1 X_{1i} + v_i$$

OVB formula: (effect of X_2 on Y) $\times$ (relationship of X_2 to X_1)

$$\hat{\beta}_1^{\text{short}} = \hat{\beta}_1^{\text{long}} + \hat{\beta}_2 \times \hat{\delta}_1$$

OVB Applied to Our Example

$$\hat{\beta}_1^{\text{short}} = \underbrace{\hat{\beta}_1^{\text{long}}}_{=0.10} + \underbrace{\hat{\beta}_2}_{\substack{\text{effect of group}\\ \text{on Performance}}} \times \underbrace{\hat{\delta}_1}_{\substack{\text{relationship of}\\ \text{group to Hours}}}$$

Varsity has higher baseline performance, so $\hat{\beta}_2 > 0$.

Scenario	$\hat{\delta}_1$	**Bias** $(\hat{\beta}_2 \times \hat{\delta}_1)$	**OLS slope**
Varsity trains more	> 0	$+$	0.22
JV trains more	< 0	$-$	-0.02
Equal training	≈ 0	≈ 0	0.10

Same data, same true effect. The OLS estimate swings from -0.02 to $+0.22$ just by changing which group trains more.

What About Class Imbalance?

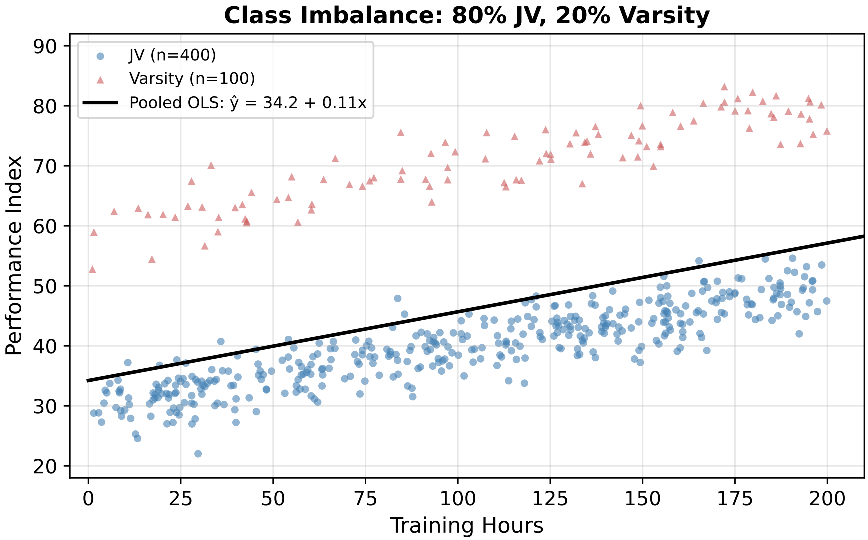
Suppose x is distributed the same across groups ($\delta \approx 0$), but the **sample composition** is unbalanced.

- Does the slope change?
- What about the intercept?

The OLS intercept is a **weighted average** of the group intercepts:

$$\begin{aligned}
\hat{\beta}_0 &= \frac{\sum_{i=1}^{n} \{i \in \text{JV}\}}{n} \cdot \beta_{0,\text{JV}} + \frac{\sum_{i=1}^{n} \{i \in \text{Var}\}}{n} \cdot \beta_{0,\text{Var}} \\
&= \frac{n_{\text{JV}}}{n} \cdot \beta_{0,\text{JV}} + \frac{n_{\text{Var}}}{n} \cdot \beta_{0,\text{Var}} \\
&= (\text{Share JV}) \cdot \beta_{0,\text{JV}} + (\text{Share Var}) \cdot \beta_{0,\text{Var}}
\end{aligned}$$

80% JV, 20% Varsity



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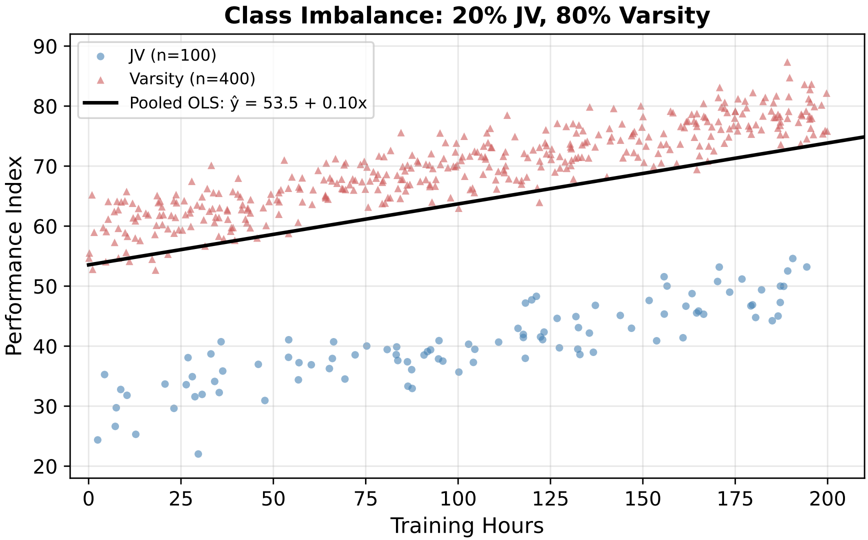


Class Imbalance: The Intercept Shifts

Sample	OLS intercept	True JV ($\alpha = 30$)	True Var ($\alpha = 60$)
80% JV	34	close	off by 26
50/50	44	off by 14	off by 16
80% Varsity	54	off by 24	close

$\Longrightarrow$ Even when the slope is approximately correct, pooled OLS uses a **single intercept** that is wrong for every subgroup. The error depends on sample composition, which the researcher may not control.

From One Intercept to Many

Every "naive" example we just saw used the same model:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

A single β_0 forces **one baseline** for all groups. That's where the bias comes from.

The fix: let each group j have its own intercept α_j, where $j =$ JV or $j =$ Varsity:

$$y_i = \alpha_j + \beta_1 x_i + \varepsilon_i$$

The subscript j on the intercept is doing all the work:

- $\beta_0 \rightarrow$ one number, shared by everyone
- $\alpha_j \rightarrow$ a **different number for each group**

This is the core idea behind **fixed effects**.

The Fixed Effects Model

Allow each group its own intercept:

$$\text{Performance}_{ij} = \alpha_j + \beta\,\text{Hours}_{ij} + \varepsilon_{ij}$$

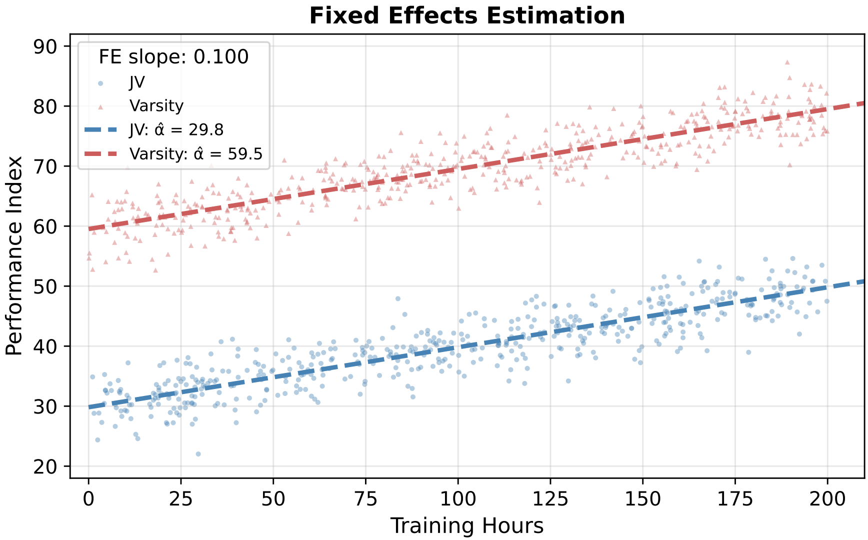
Equivalently, add a group dummy:

$$y_i = \beta\,x_i + \alpha_{\text{JV}} \cdot \{\text{JV}\} + \alpha_{\text{Var}} \cdot \{\text{Var}\} + \varepsilon_i$$

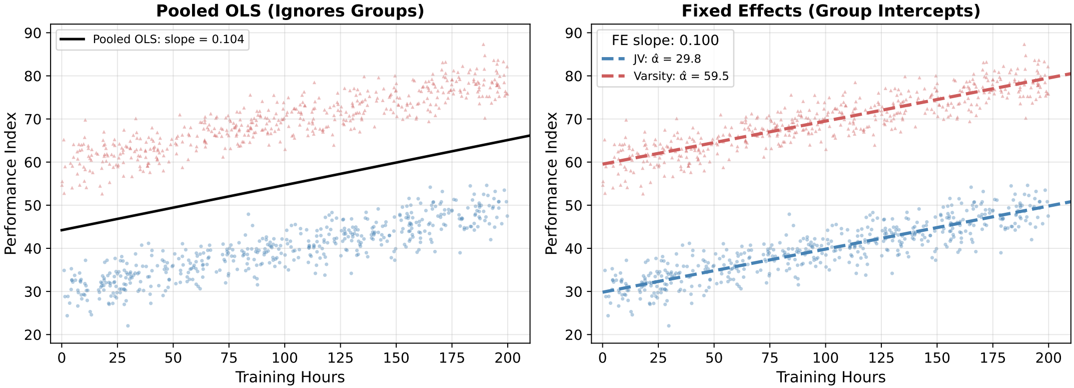
Key idea: FE estimates β using only *within-group* variation in x.

It asks: "Among JV players, do those who train more perform better?"

FE Estimation: The Result



Pooled OLS vs. Fixed Effects



Why Does This Work? The Rank Condition

Basic idea: You need at least **two points to draw a line** (one intercept + one slope = 2 unknowns).

Pooled OLS: 2 unknowns (β_0, β_1) $\Longrightarrow$ need ≥ 2 observations total. Easy.

Fixed effects with 2 groups: 3 unknowns (α_{JV}, α_{Var}, β_1).

- You need ≥ 2 observations *per group* (to pin down each group's line)
- So at minimum: **4 observations** (2 JV + 2 Varsity)

In general: with J groups, you have $J + 1$ unknowns (J intercepts + 1 slope), so you need at least 2 observations per group to be identified.

$\Longrightarrow$ More groups = more unknowns = more data required.

Connection to Panel Data

Our training example maps directly to the panel data framework:

$$y_{it} = \alpha_i + \beta\, x_{it} + \varepsilon_{it}$$

- i = individual (trainee $\rightarrow$ firm, person, country)
- t = time period
- α_i = **individual fixed effect** (unobserved, time-invariant)

Two equivalent estimation approaches:

1. **Least Squares Dummy Variable** (when we want all of the individual fixed effects):

$$y_{it} = \beta\, x_{it} + \sum_i \alpha_i D_i + \varepsilon_{it}$$

2. **Within / Demeaning Estimator:** Subtract individual means

$$(y_{it} - \bar{y}_i) = \beta(x_{it} - \bar{x}_i) + (\varepsilon_{it} - \bar{\varepsilon}_i)$$

Both give the **same** $\hat{\beta}$. Let's walk through the within estimator step by step.

The Within Estimator: Step 1 $\rightarrow$ Compute Group Means

Start with the model:

$$y_{it} = \alpha_i + \beta\, x_{it} + \varepsilon_{it}$$

For each individual i, take the **time average** of both sides:

$$\bar{y}_i = \alpha_i + \beta\, \bar{x}_i + \bar{\varepsilon}_i$$

where:

- $\bar{y}_i = \frac{1}{T}\sum_{t=1}^{T} y_{it}$ (mean outcome for individual i)
- $\bar{x}_i = \frac{1}{T}\sum_{t=1}^{T} x_{it}$ (mean regressor for individual i)

Notice that α_i survives averaging because it **doesn't vary over time**.

The Within Estimator: Step 2 $\rightarrow$ Subtract

Subtract the individual mean equation from the original:

$$y_{it} - \bar{y}_i = (\alpha_i - \alpha_i) + \beta(x_{it} - \bar{x}_i) + (\varepsilon_{it} - \bar{\varepsilon}_i)$$

The fixed effect cancels:

$$\ddot{y}_{it} = \beta\,\ddot{x}_{it} + \ddot{\varepsilon}_{it}$$

where $\ddot{y}_{it} \equiv y_{it} - \bar{y}_i$ is the **demeaned** variable.

$\implies$ We have a simple regression with **no intercept** and no α_i.
Just run OLS on the demeaned data.

The Within Estimator: Step 3 $\rightarrow$ Estimate β

OLS on the demeaned regression gives:

$$\hat{\beta} = \frac{\text{Cov}(\ddot{x}_{it}, \ddot{y}_{it})}{\text{Var}(\ddot{x}_{it})} = \frac{\sum_i \sum_t (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i)}{\sum_i \sum_t (x_{it} - \bar{x}_i)^2}$$

$\implies$ The covariance of the demeaned variables equals the covariance of the original variables:

$$\text{Cov}(\ddot{x}_{it}, \ddot{y}_{it}) = \text{Cov}(x_{it} - \bar{x}_i, y_{it} - \bar{y}_i) = \text{Cov}(x_{it}, y_{it})$$

Why? Let's show it briefly.

Proof: Demeaning Doesn't Change Covariance

Let $c_i = \bar{x}_i$ and $d_i = \bar{y}_i$ (constants within group i). Then:

$$\begin{aligned}\text{Cov}(x_{it} - c_i,\ y_{it} - d_i) = \text{Cov}(x_{it},\ y_{it}) - \text{Cov}(x_{it},\ d_i) \\ - \text{Cov}(c_i,\ y_{it}) + \text{Cov}(c_i,\ d_i)\end{aligned}$$

Within each group i, the means c_i and d_i are **constants**, so:

$$\text{Cov}(x_{it},\ d_i) = 0, \quad \text{Cov}(c_i,\ y_{it}) = 0, \quad \text{Cov}(c_i,\ d_i) = 0$$

Therefore:

$$\text{Cov}(x_{it} - \bar{x}_i,\ y_{it} - \bar{y}_i) = \text{Cov}(x_{it},\ y_{it}) \qquad \blacksquare$$

$\Longrightarrow$ The within estimator uses **only within-group variation**. All between-group differences are "absorbed" by the fixed effects.

Thank you!

jakeanderson@g.ucla.edu