Introduction to Heteroskedasticity

Non-Constant Variance and What to Do About It

Jake Anderson

March 3, 2026

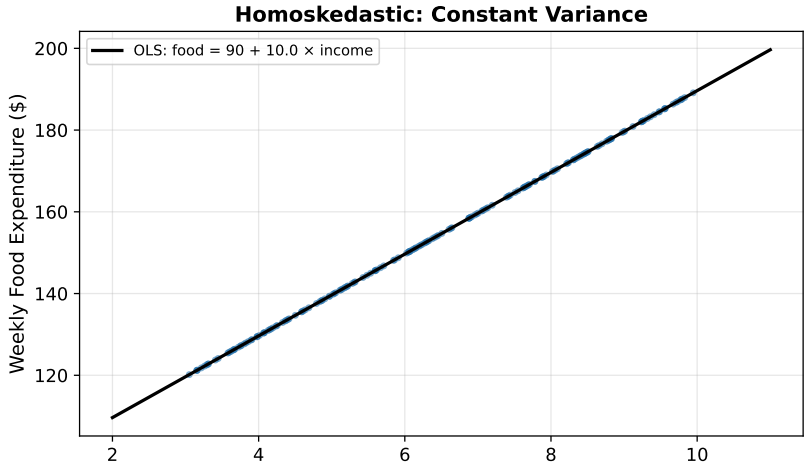
Outline

1. Motivation
2. What Goes Wrong
3. Detecting Heteroskedasticity
4. Fixing It: Robust Standard Errors
5. Fixing It: WLS
6. Fixing It: FGLS
7. Worked Example: Food Expenditure (HGL 8.1)

Food Expenditure vs. Income: The Simple Case

Suppose we model weekly food expenditure as a function of household income:

$$\text{food}_i = \beta_0 + \beta_1 \, \text{income}_i + e_i$$



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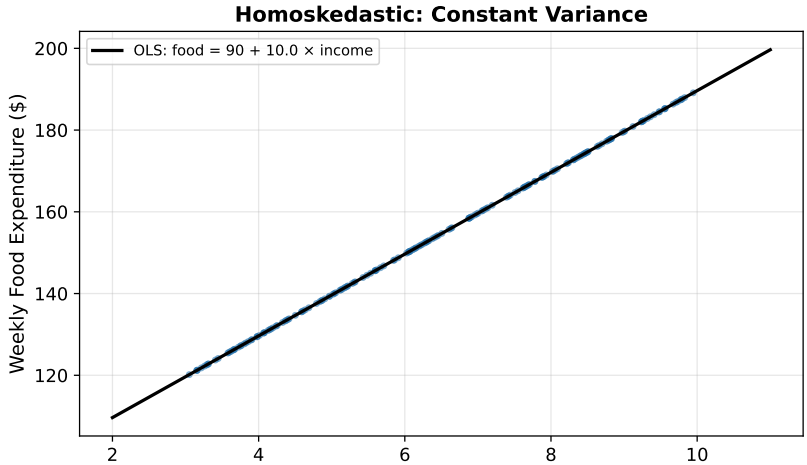

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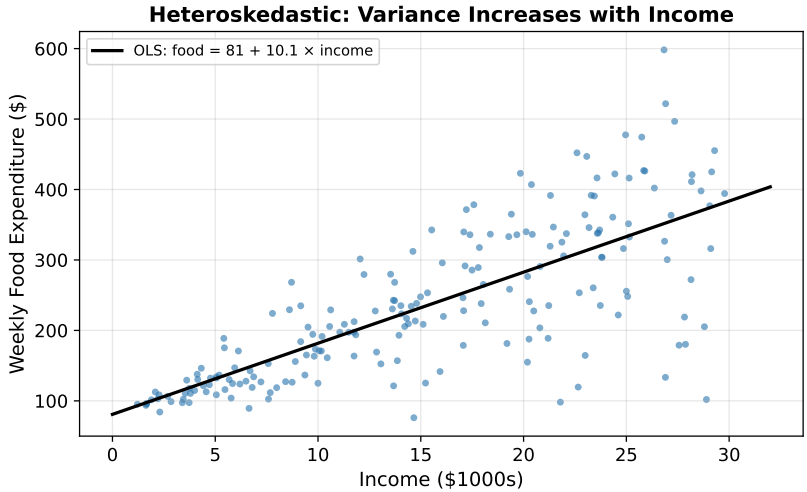

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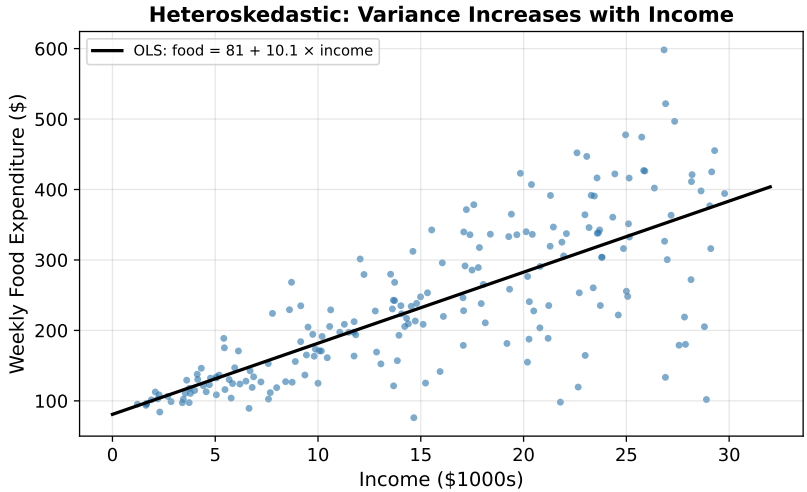

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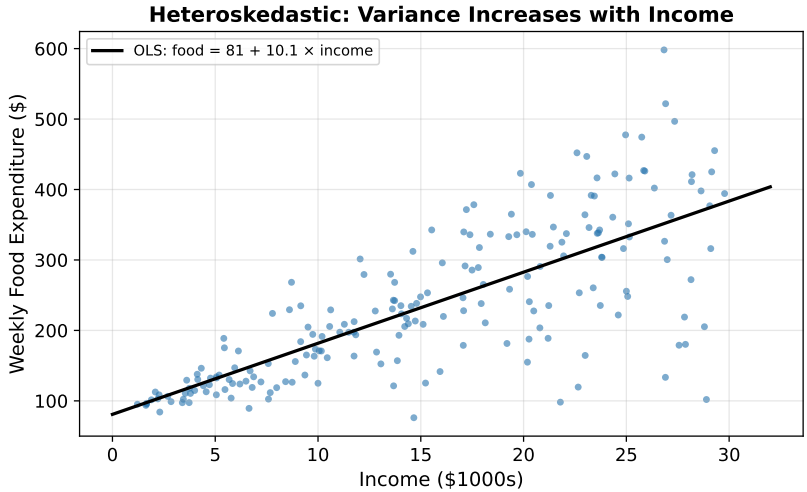

Why Does Variance Change?

Heteroskedasticity is common whenever:

- **Scale effects:** Higher income $\implies$ more discretion in spending
- **Learning:** Experienced firms have tighter cost control than new ones
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Formally, we write:

$$\text{Var}(e_i \mid x_i) = \sigma_i^2$$

where σ_i^2 depends on x_i (or some other observable). The assumption $\sigma_i^2 = \sigma^2$ for all i is what we call **homoskedasticity**.

OLS Under Heteroskedasticity: The Good News

Even with heteroskedasticity, OLS is still:

- **Unbiased:** $\mathrm{E}[\hat{\beta}] = \beta$ (requires only $\mathrm{E}[e_i \mid x_i] = 0$)
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$\Longrightarrow$ The coefficient estimates themselves are fine. The problem is elsewhere.

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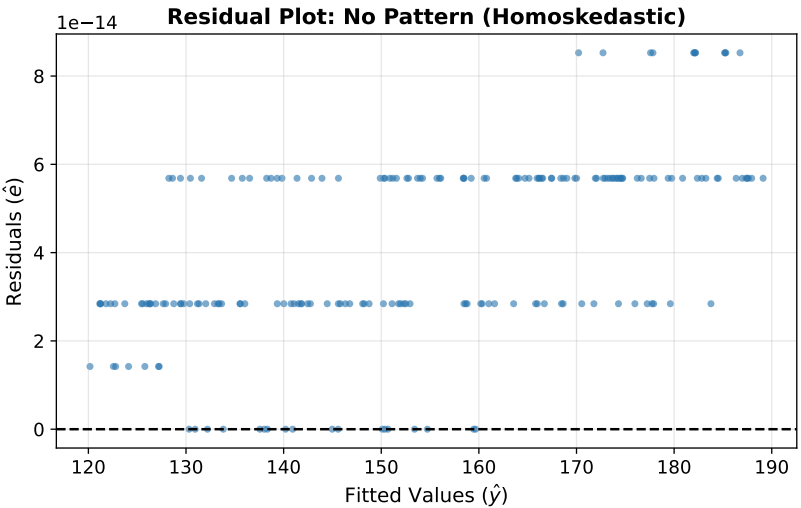
$\Longrightarrow$ You cannot trust hypothesis tests from OLS when heteroskedasticity is present.

Visual Detection: Residual Plots

Plot residuals $\hat{e}_i$ against fitted values $\hat{y}_i$ (or against x_i):



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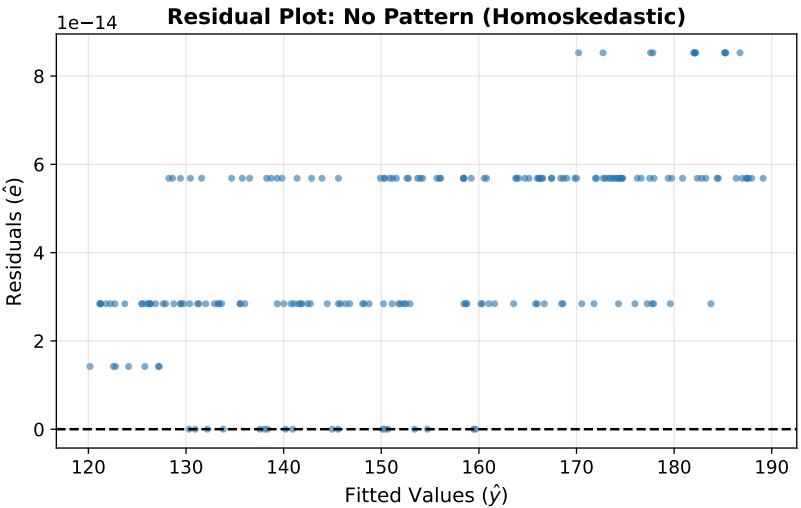

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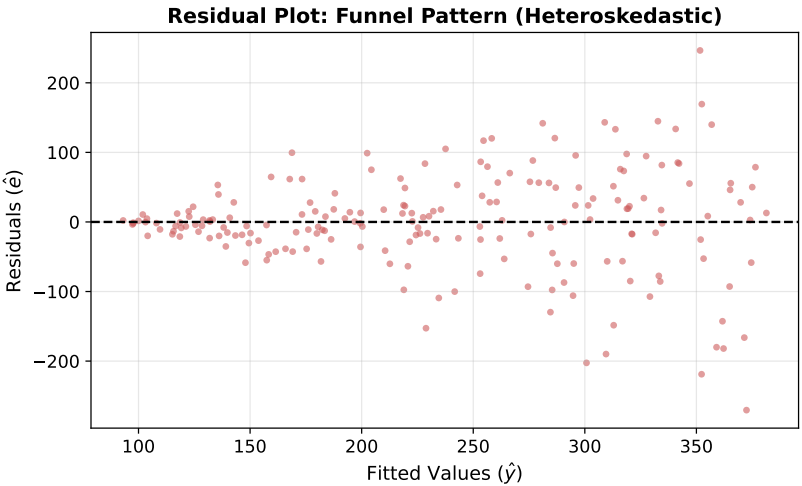

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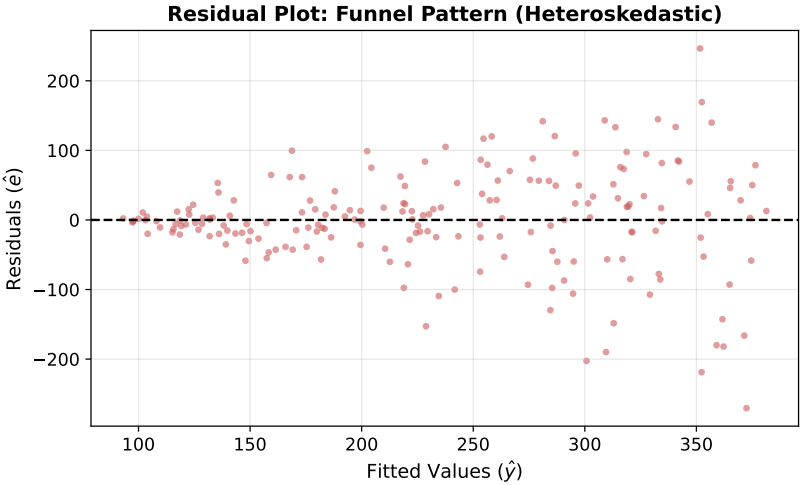

Residuals fan out as $\hat{y}$ increases $\implies$ strong evidence of heteroskedasticity. This "funnel" or

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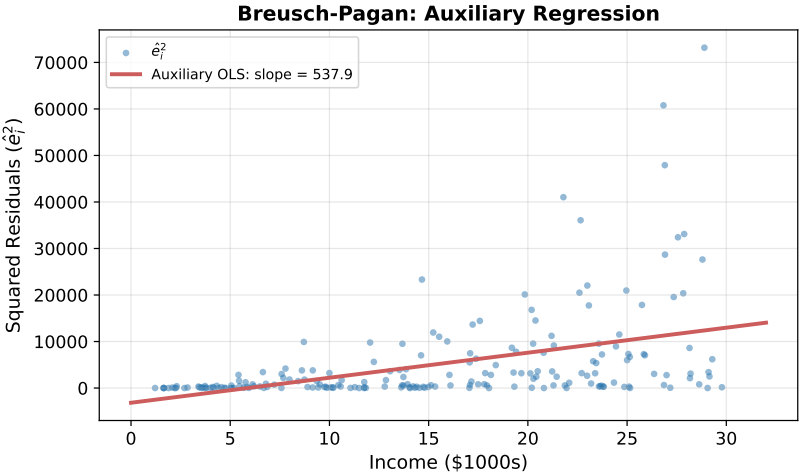
With multiple regressors: include all x's in the auxiliary regression. Then $\text{BP} \sim \chi^2(k)$ where $k =$ number of regressors.

BP Test: The Auxiliary Regression



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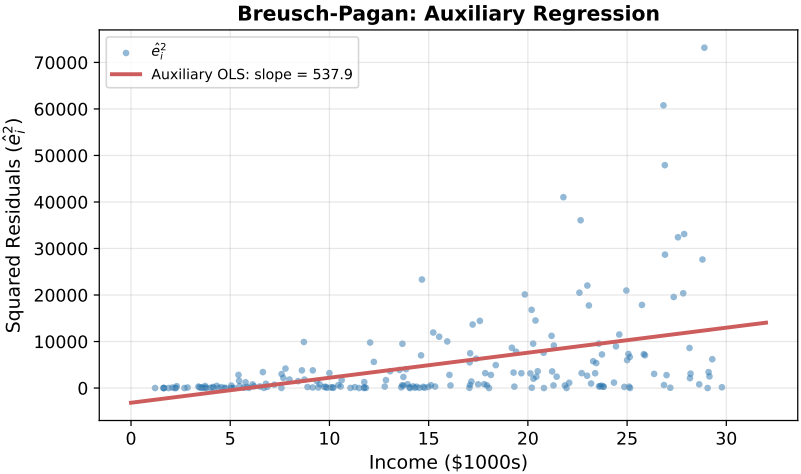

The auxiliary regression line slopes upward $\Longrightarrow$ variance increases with income. A flat line would indicate homoskedasticity.

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Advantage: detects **nonlinear** forms of heteroskedasticity.
Disadvantage: with many regressors, the auxiliary regression has many terms $\implies$ low power.

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$\Longrightarrow$ Simple and intuitive, but requires choosing which variable to sort by.

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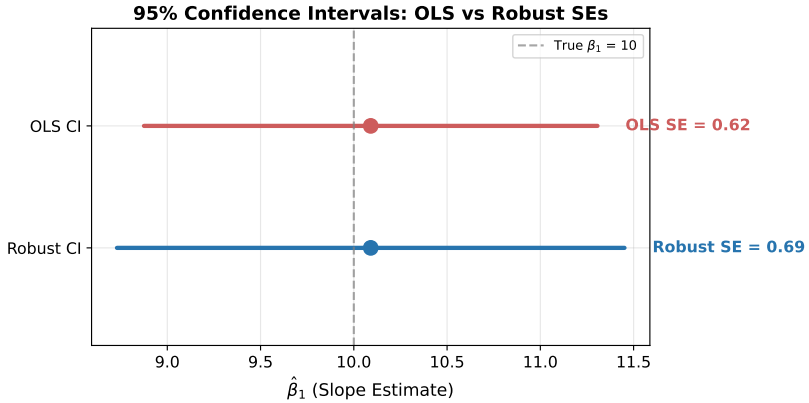
HC1 (small-sample correction): multiply by $\dfrac{n}{n-k}$

HC1 is the default "robust" SE in most software. With large n, HC0 and HC1 are nearly identical.

Robust SEs: What Changes?



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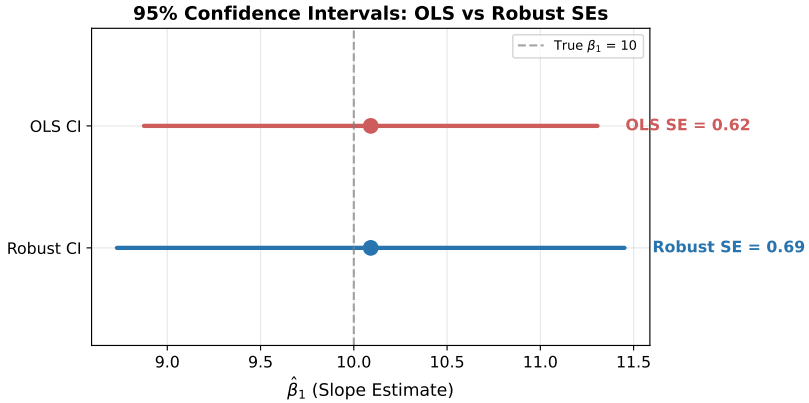

- The **point estimate** $\hat{\beta}_1$ does not change (same OLS regression)
- Only the **standard error** (and therefore the CI width) changes
- With heteroskedasticity, OLS SEs are typically **too small** $\implies$ robust CIs are wider

Robust SEs in R

Using the car and lmtest packages:

```
library(car) library(lmtest)
model <- lm(food   income, data = food_data)
# Usual OLS standard errors summary(model)
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$\Longrightarrow$ Always report robust SEs unless you have strong reason to believe homoskedasticity holds. Many applied researchers use robust SEs by default.

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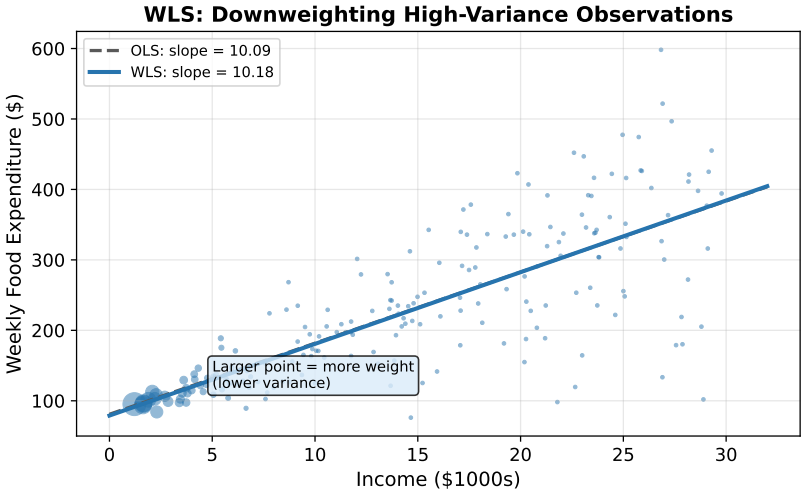
WLS minimizes:

$$\sum_i w_i(y_i - \beta_0 - \beta_1 x_i)^2 = \sum_i \frac{(y_i - \beta_0 - \beta_1 x_i)^2}{\sigma_i^2}$$

WLS: Visual Intuition



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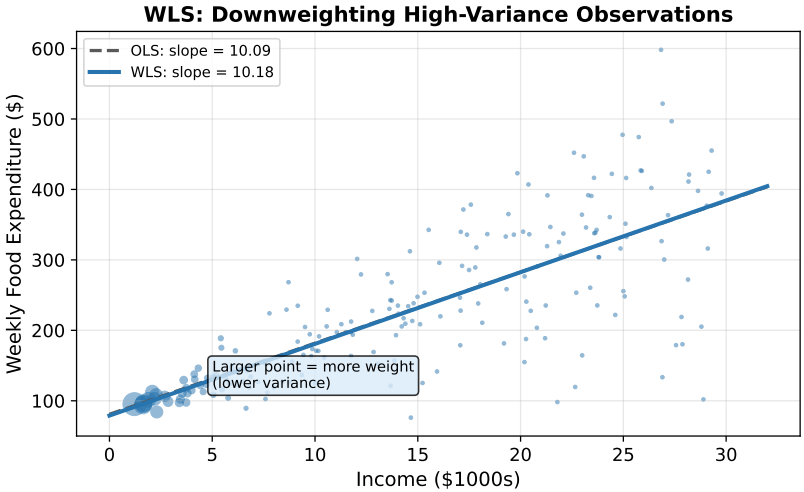

Larger points have more weight (lower variance). The WLS line is pulled toward the low-income,

WLS as a Transformation

Divide the entire equation by σ_i:

$$\frac{y_i}{\sigma_i} = \beta_0 \cdot \frac{1}{\sigma_i} + \beta_1 \cdot \frac{x_i}{\sigma_i} + \frac{e_i}{\sigma_i}$$

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WLS is **BLUE** when the weights are correct. It is more efficient than OLS.

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R note: The `weights` argument in `lm()` expects $w_i = 1/\sigma_i^2$, not σ_i^2:

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# If Var(e_i) proportional to income: wls_model <- lm(food   income, data = food_data, weights = 1 / incor
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WLS: The Catch

WLS requires knowing σ_i^2 (or at least σ_i^2 up to a proportionality constant).

Common assumptions:

- $\text{Var}(e_i) = \sigma^2 x_i \implies w_i = 1/x_i$
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$\implies$ If we know the form of heteroskedasticity, WLS gives us efficient estimates. But what if we don't know σ_i^2?

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When σ_i^2 is unknown, we **estimate** it from the data in two steps.

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This estimates how variance scales with x.

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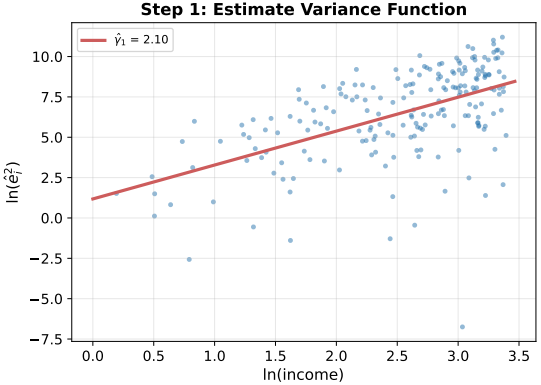
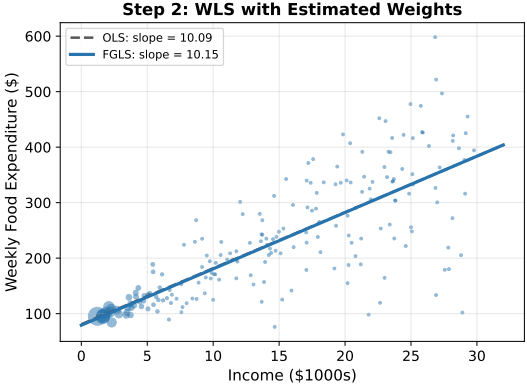
$\Longrightarrow$ FGLS is "feasible" because it replaces the unknown σ_i^2 with an estimate $\hat{\sigma}_i^2$. With large n, FGLS approaches the efficiency of true GLS.

FGLS: The Two Steps





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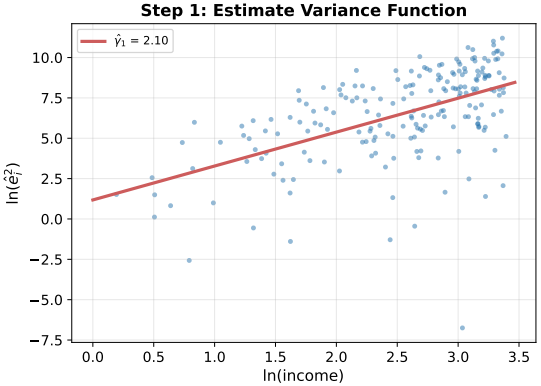

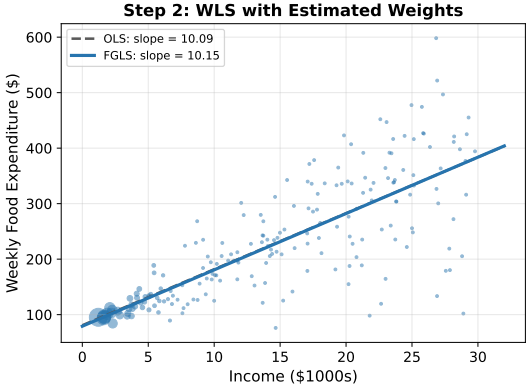

Left: the auxiliary regression reveals how variance scales with income. Right: FGLS uses these estimated weights to refit the model.

Exercise 8.1: Setup

Model: $\text{food}_i = \beta_1 + \beta_2 \, \text{income}_i + e_i$

Data: 40 households. Weekly food expenditure ($) and weekly income ($100s).

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$$\widehat{\text{food}} = 83.42 + 10.21 \, \text{income}$$

	$\hat{\beta}_1$ (intercept)	$\hat{\beta}_2$ (income)
Estimate	83.42	10.21
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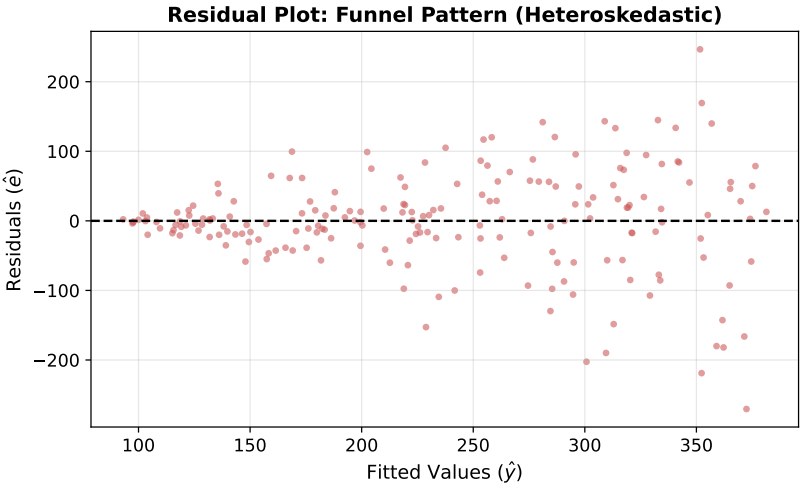
$t = 10.21/2.09 = 4.88 \implies$ significant at any conventional level. But can we trust this SE?

Residual Plot



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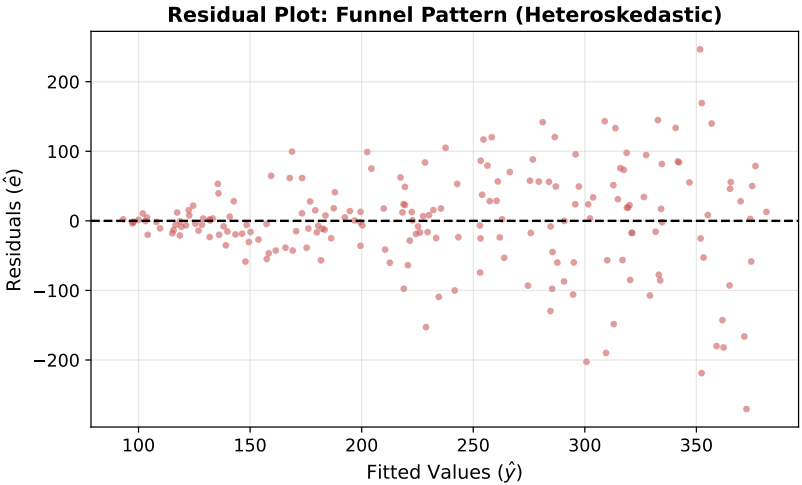

Clear funnel pattern $\Longrightarrow$ heteroskedasticity. The OLS SE of 2.09 is suspect.

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If $\text{GQ} > F_{0.05}(14, 14) = 2.48$ $\implies$ reject homoskedasticity.

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$\implies$ The GQ test is intuitive: does the high-income group have a bigger residual sum of squares than the low-income group?

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$\implies$ In this case, OLS SEs understate uncertainty. Robust SEs give honest inference.

Computing Robust CIs

95% CI formula (same as always, just swap in the robust SE):

$$\hat{\beta}_2 \pm t_{0.025,\, n-2} \times \mathrm{SE}_{\mathrm{robust}}(\hat{\beta}_2)$$

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In R:

```
library(car); library(lmtest) model <- lm(food   income, data = food_data) # Robust t-test and CI coeftes
vcov. = hccm(model, type = "hc1")) confint(coeftest(model, vcov. = hccm(model, type = "hc1")))
```

Summary: Which Fix to Use?

Method	**When to Use**	**Limitation**
Robust SEs	Always safe; default in applied work	Does not improve efficiency
WLS	You know $\text{Var}(e_i) \propto h(x_i)$	Requires correct variance form
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$\Longrightarrow$ Robust SEs are the minimum standard. WLS/FGLS can do better if you model the variance correctly.

Thank you!
jakeanderson@g.ucla.edu