Measurement Error

Attenuation Bias and the Reliability Ratio

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Motivation: Study Time and Office Hours

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$\implies$ Office hours attendance is a **noisy** version of the true variable we care about.

Why Measurement Error Causes Endogeneity

Substitute $x_i^* = x_i - u_i$ into the true model:

$$y_i = \beta_1 + \beta_2(x_i - u_i) + v_i = \beta_1 + \beta_2 x_i + \underbrace{(v_i - \beta_2 u_i)}_{e_i}$$

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\text{Cov}(x_i, e_i) &= \text{Cov}(x_i^* + u_i,\ v_i - \beta_2 u_i) \\
&= \underbrace{\text{Cov}(x_i^*, v_i)}_{=0} - \beta_2 \underbrace{\text{Cov}(x_i^*, u_i)}_{=0} + \underbrace{\text{Cov}(u_i, v_i)}_{=0} - \beta_2 \underbrace{\text{Cov}(u_i, u_i)}_{=\sigma_u^2} \\
&= -\beta_2 \sigma_u^2 \neq 0
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$\Longrightarrow$ If $\beta_2 > 0$, there is a **negative** correlation between x_i and e_i. OLS underestimates β_2.

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Attenuation Bias Formula

As $N \to \infty$, the OLS estimator converges to:

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- If $\sigma^2_u = 0$ (no error): $\lambda = 1$ and $b_2 \to \beta_2$ (no bias)
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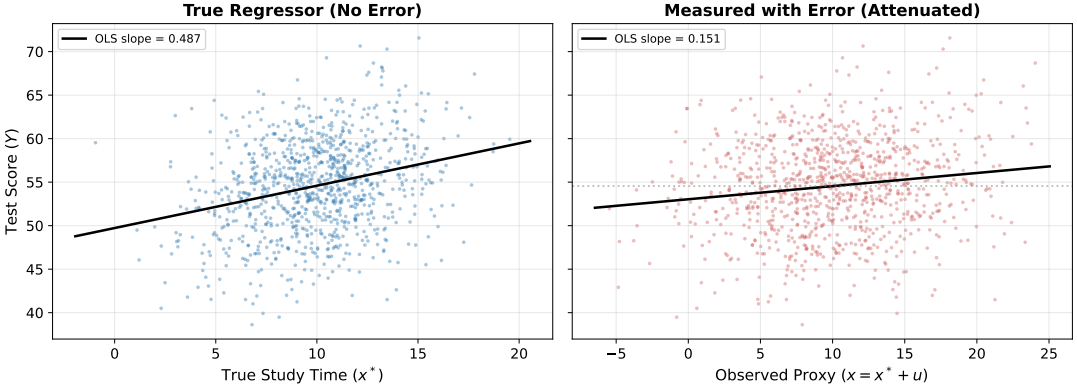
$\implies$ Measurement error **always shrinks the coefficient toward zero**. This is called **attenuation bias**. More data does not help.

Visualizing Measurement Error



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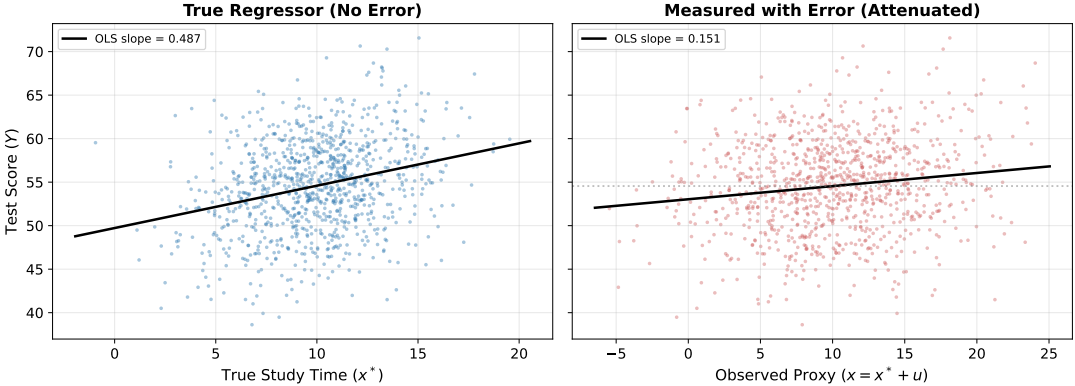
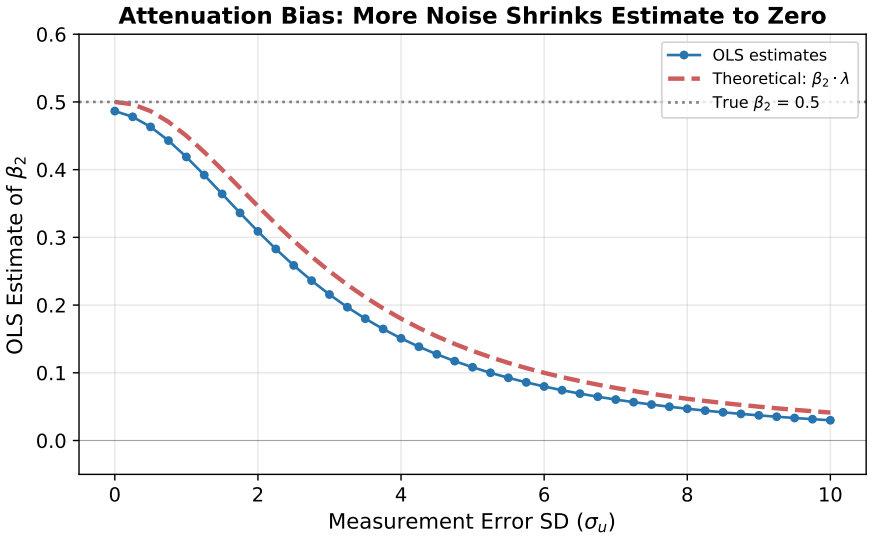


Left: using true x^*, OLS recovers the correct slope. Right: using observed x, the scatter is wider and the slope is attenuated.

Simulation: Attenuation Bias in Action



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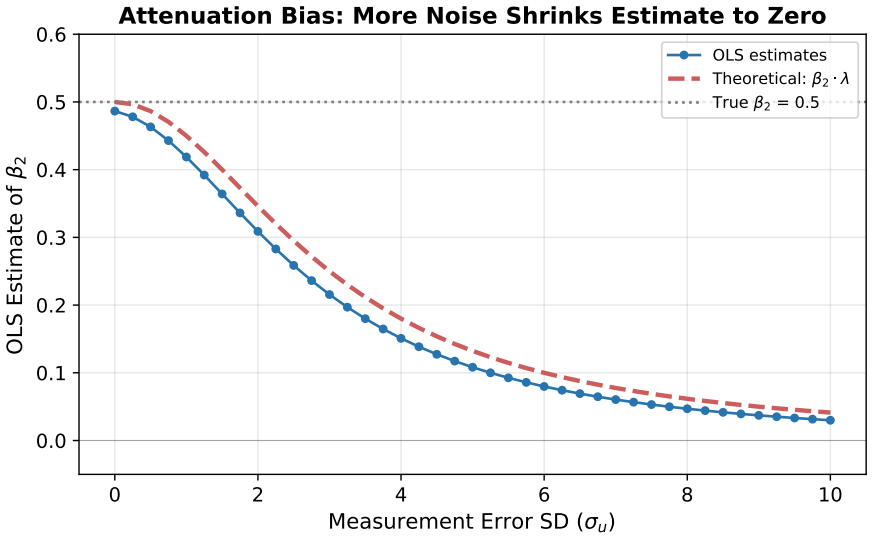
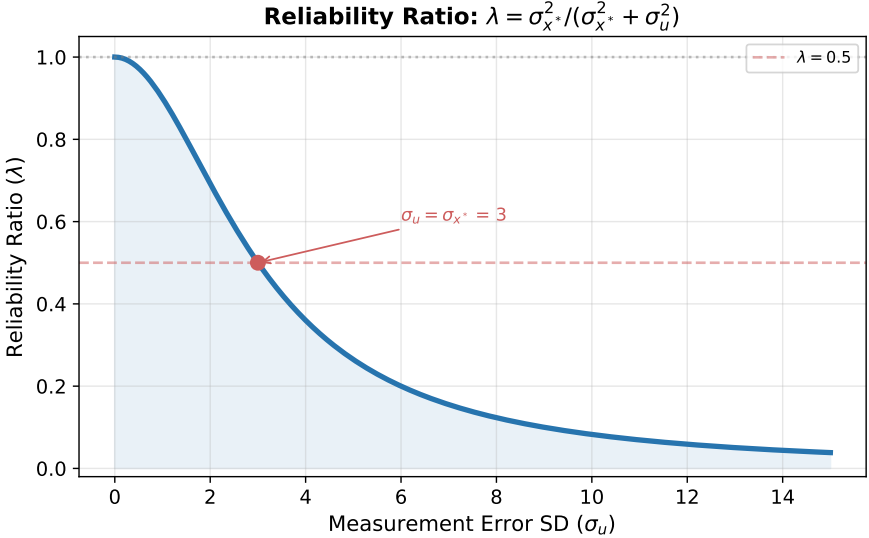


The Reliability Ratio



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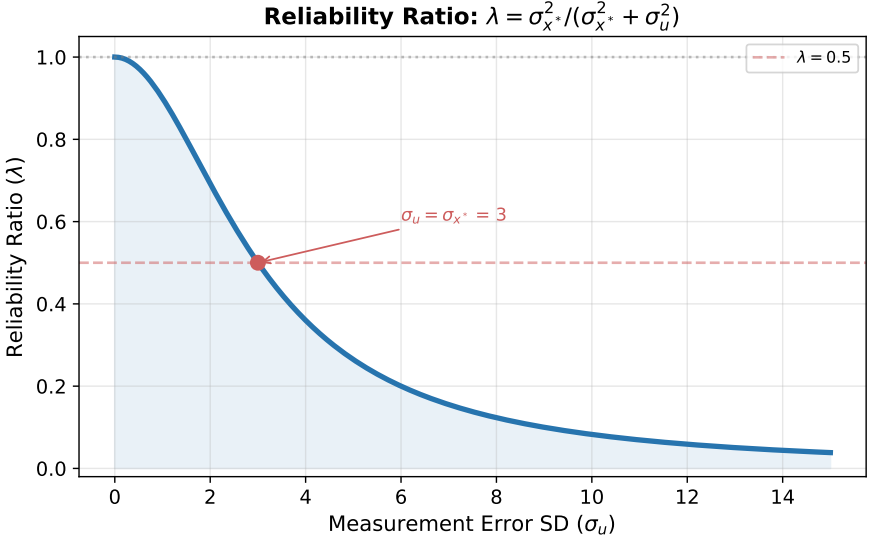

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- More data does **not** fix it: the bias persists even as $N \to \infty$
- Solutions: find better measures of the true variable, or use instrumental variables

Thank you!
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