

Multinomial Logit / Conditional Logit

Why You Can't Just Run Four Logits

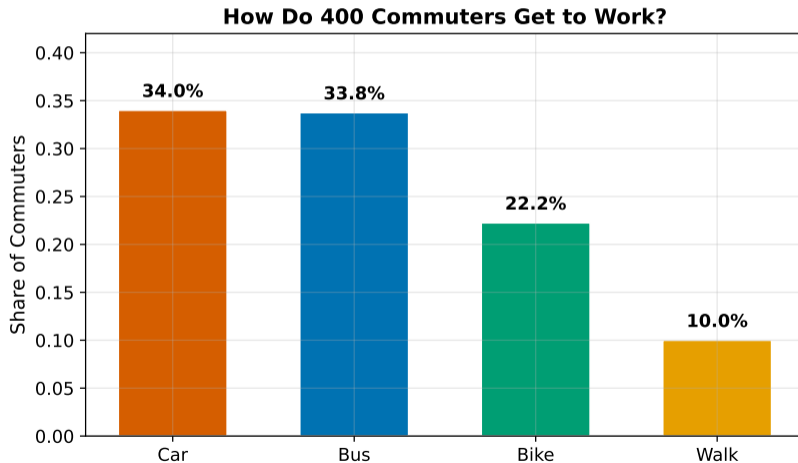
Jake Anderson

March 3, 2026

- 1 The Problem: More Than Two Choices
- 2 Multinomial Logit
- 3 Conditional Logit
- 4 Independence of Irrelevant Alternatives (IIA)

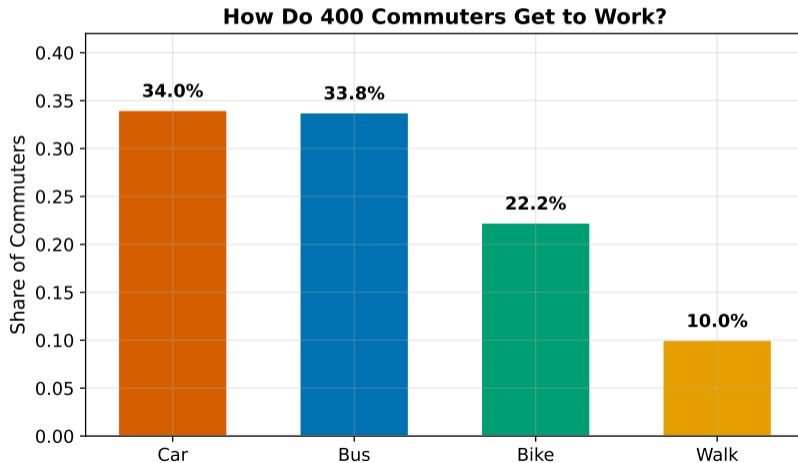
The Data

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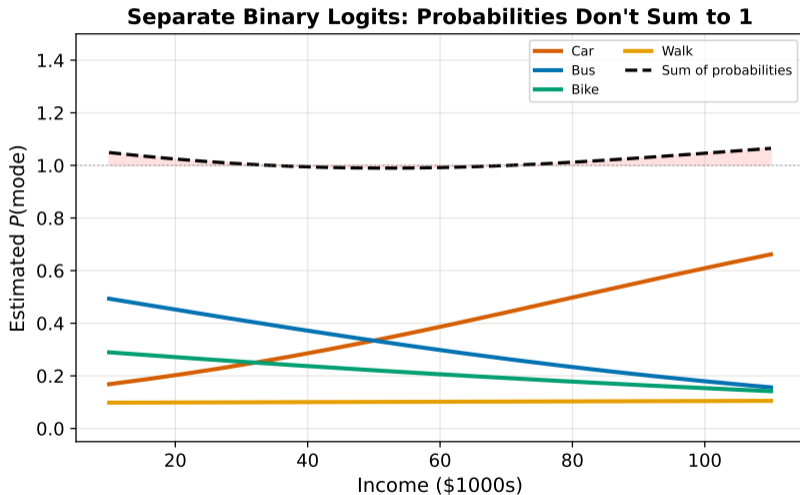
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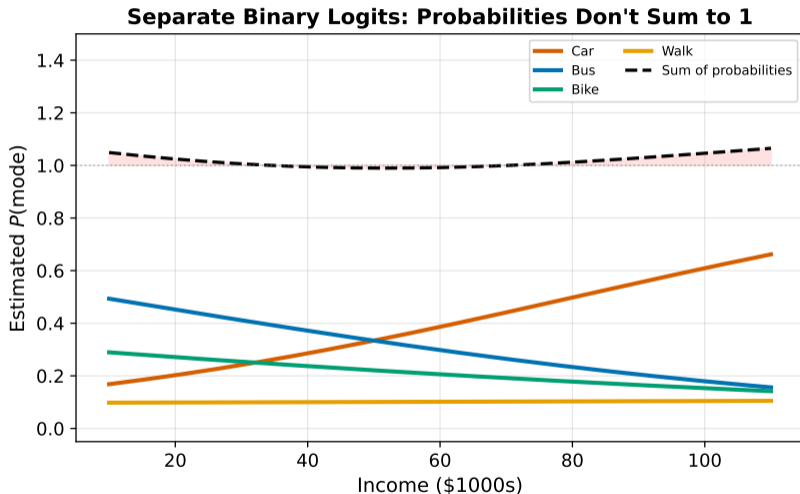
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But there is a structural problem: four separate models know nothing about each other.

Separate Binary Logits: The Failure

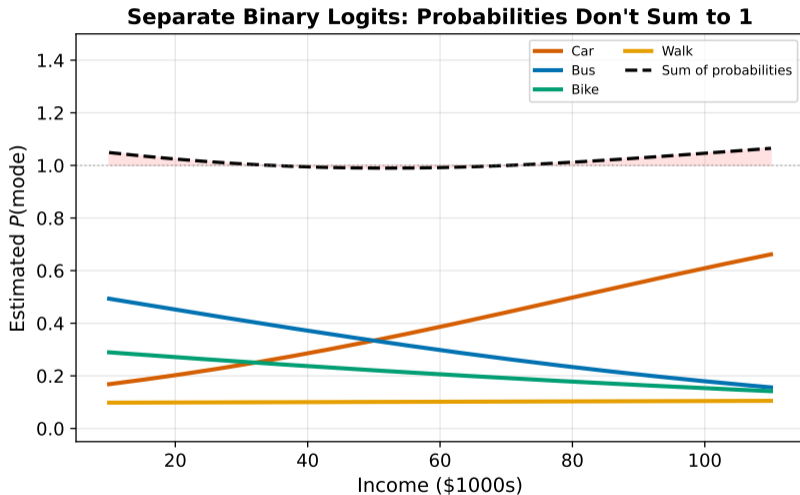


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\implies The multinomial logit model achieves exactly this.

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⇒ We never observe utility directly. We observe the *choice*, which reveals which mode had the highest utility for that person.

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\implies The softmax forces probabilities to be positive and sum to 1. This is the structural constraint that separate binary logits violate.

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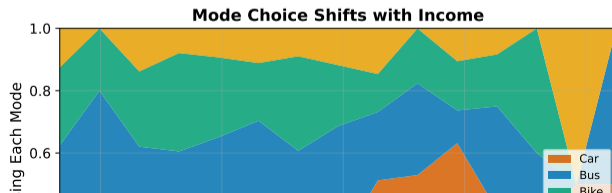
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Higher income \implies more likely to drive. Longer bus time \implies less likely to take the bus.



MNL with Individual-Specific Variables

When the regressors vary across *people* but not across alternatives (e.g., income), the systematic utility is:

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⇒ The same variable (income) has a *different coefficient for each alternative*. This is what makes the probability curves fan out differently.

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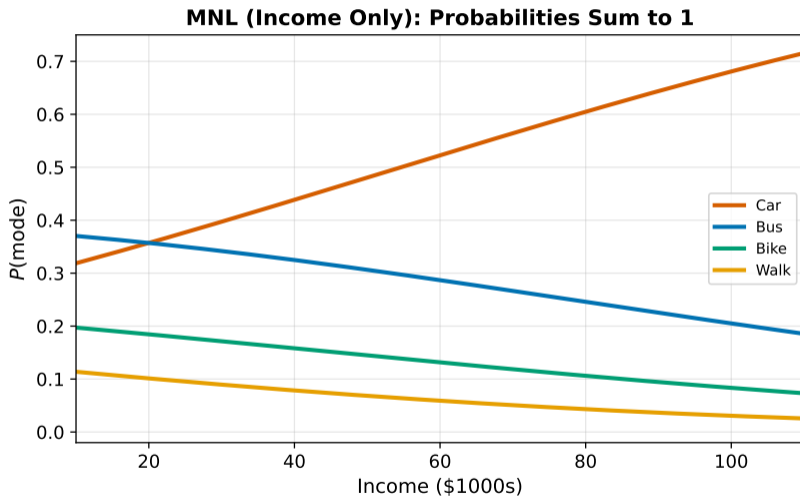
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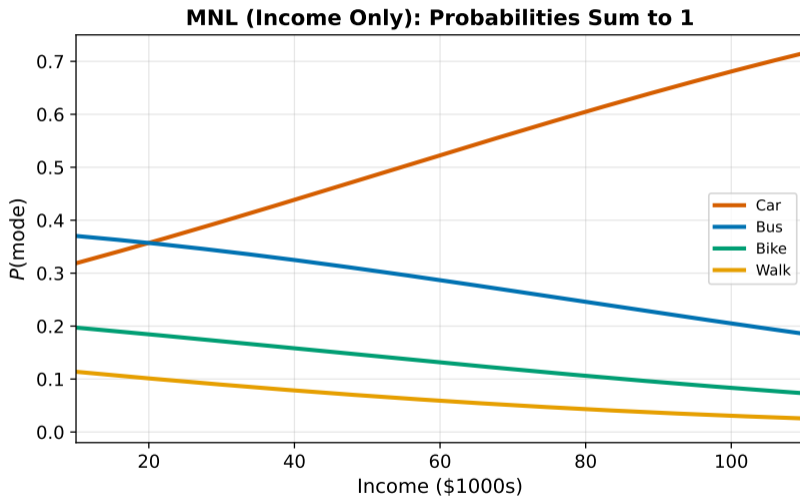
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\implies Car has a lower baseline than Bus ($\alpha_{\text{Car}} < 0$), but the positive income effect ($\beta_{\text{Car}} = 0.15$) makes Car most likely at this income level.



MNL Probabilities: Visualized



As income rises, Car probability increases while Bike and Walk decline. At every income level, the four

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All estimated coefficients are then interpreted *relative to Bus*:

- $\beta_{\text{Car}} > 0$: higher income increases Car utility *relative to Bus*

Interpreting Coefficients

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The sign of β_j tells us the direction *relative to Bus*. It does not directly give the marginal effect on probability (same nonlinearity issue as binary logit).

Marginal Effects in the Multinomial Logit

For an individual-specific variable x_i , the marginal effect on $P(y_i = j)$ is:

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⇒ Report Average Marginal Effects (AMEs) for each alternative, just as in binary logit.

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No closed-form solution \implies solved numerically, just like binary logit. Software handles this automatically.

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⇒ We need to incorporate **alternative-specific variables**: attributes that vary across both people *and* modes (travel time, travel cost).

Conditional Logit: Alternative-Specific Variables

Conditional logit (McFadden, 1974) enters alternative-specific variables with a **single coefficient** shared across all modes:

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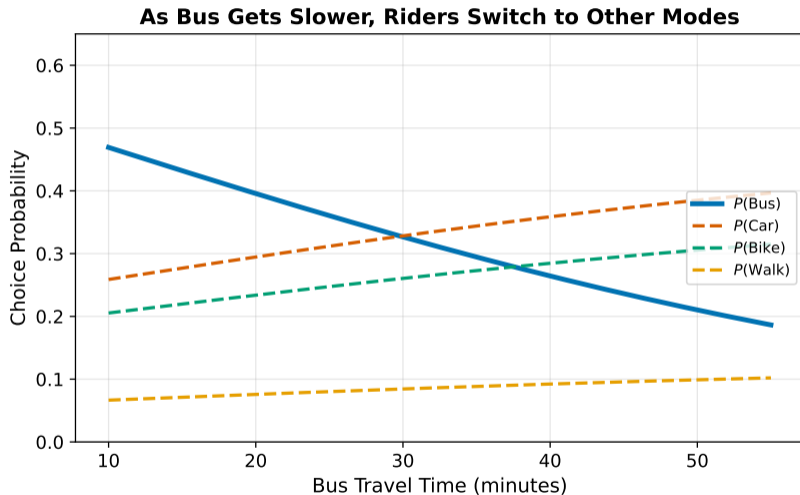
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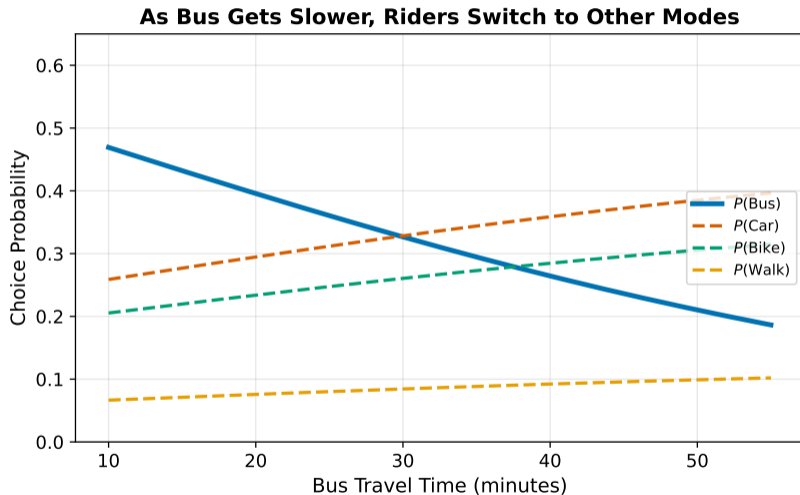
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⇒ One extra minute of travel time reduces utility by β_{time} , regardless of whether it is a minute on the bus or a minute in the car.



Conditional Logit: Travel Time Effect



As bus travel time increases, $P(\text{Bus})$ falls and the other modes absorb the lost share. The rate of

The Full Model: Combining Both Variable Types

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⇒ This combined model is often called the “conditional logit” or “McFadden’s choice model” in applied work, though terminology varies across textbooks. It handles both types of variation simultaneously.

Model Comparison: What Each Specification Handles

	MNL	CL	Combined
Individual-specific variables (income)	✓		✓
Alternative-specific variables (time, cost)		✓	✓
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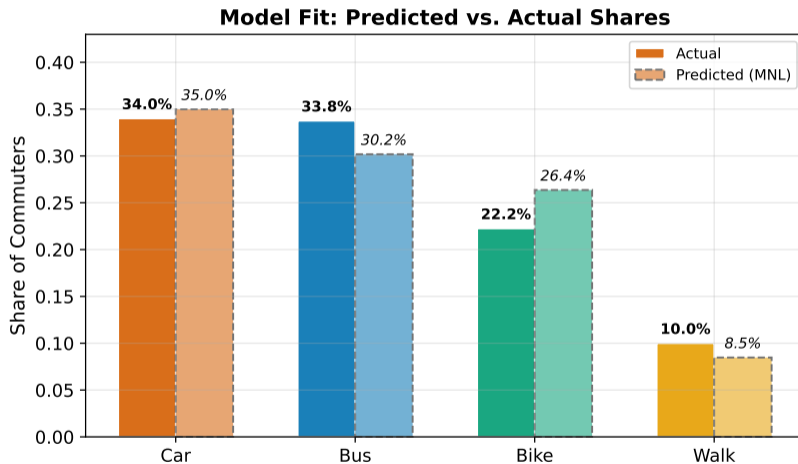
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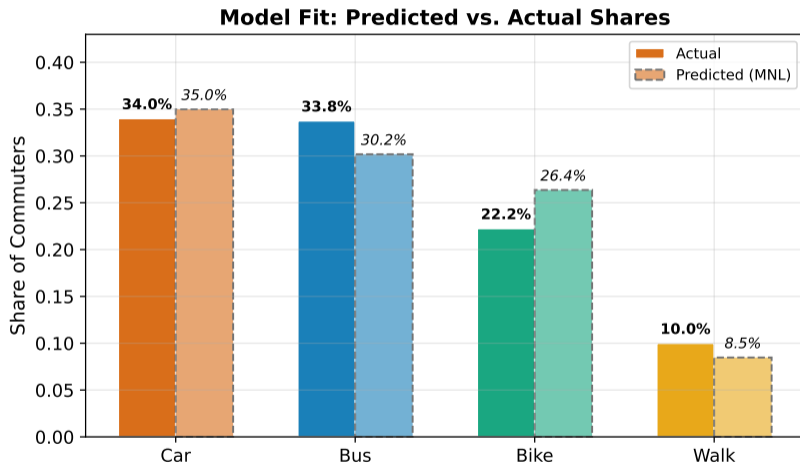
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All three use the same softmax probability and the same MLE estimator. The only difference is what goes into V_{ij} .

Model Fit: Predicted vs. Actual



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The multinomial/conditional logit closely matches the observed mode shares. The model aggregates well even though individual predictions are probabilistic.

- 1 The Problem: More Than Two Choices
- 2 Multinomial Logit
- 3 Conditional Logit
- 4 Independence of Irrelevant Alternatives (IIA)

The multinomial logit probability has a special structure. Take the ratio of any two alternatives:

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⇒ **Independence of Irrelevant Alternatives (IIA)**: the relative odds between any two modes are unaffected by what other modes exist.

This is both a strength (clean, tractable) and a weakness (unrealistic in some settings). Note: this property holds given fixed parameters, which is the standard presentation of IIA.

The Red Bus / Blue Bus Problem

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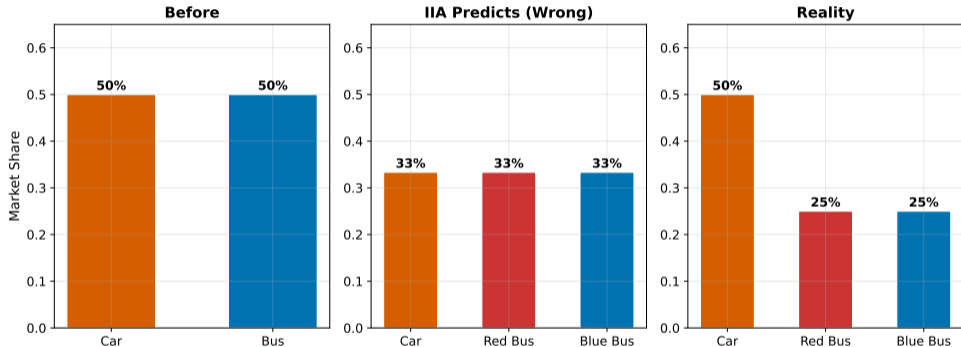
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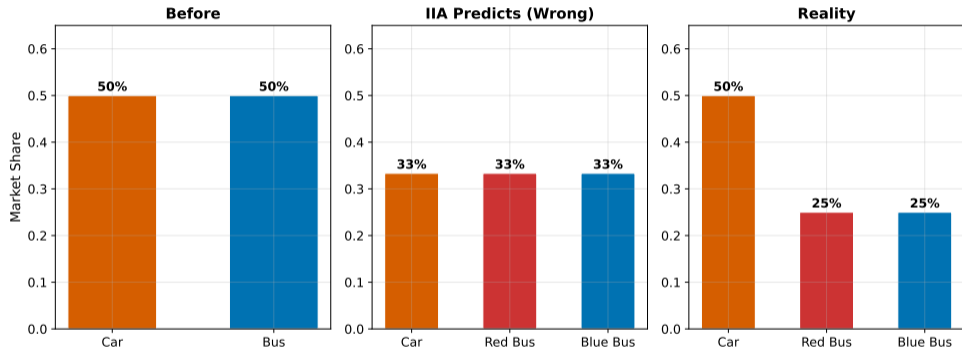
But common sense says bus riders split between Red and Blue, while car drivers are unaffected:

- Car: 50%, Red Bus: 25%, Blue Bus: 25%

Red Bus / Blue Bus: Visualized



Red Bus / Blue Bus: Visualized



IIA says the relative odds of Car to Blue Bus stay the same after Red Bus enters. In reality, Red Bus steals from Blue Bus (a close substitute), not proportionally from all modes.

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Hausman test for IIA: estimate the model on a subset of alternatives. If IIA holds, the coefficients should not change significantly when you drop one alternative.

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Model	How it relaxes IIA
Nested logit	Groups similar alternatives into “nests” (e.g., Motorized: {Car, Bus} vs. Non-motorized: {Bike, Walk}). Allows correlation within nests
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⇒ Start with multinomial logit and test IIA. If it fails, move to nested or mixed logit.

- ① **Only individual-specific variables?** (income, age, etc.)
⇒ Multinomial logit (MNL)

Decision Framework: Which Model to Use

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⇒ In most applied work, start with the combined model. It nests the other two as special cases.

Thank you!
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