Simultaneous Equations

The Chicken and the Egg

Jake Anderson

March 3, 2026

"My belt holds my pants up, but the belt loops hold my belt up.

I don't really know what's happening down there.

Who is the real hero?"

— Mitch Hedberg

Outline

The Problem: Jointly Determined Variables

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$\Longrightarrow$ Regressing Q on P with OLS does not recover the demand curve or the supply curve. It produces a confused mixture of both.

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2	Hot week (demand ↑)	\$6.00	140
3	Pearl shortage (supply ↓)	\$6.00	80
4	Instagram viral (demand ↑)	\$5.60	128
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$\Longrightarrow$ A single OLS line through (P_t, Q_t) cannot recover demand or supply.

Bruin Boba: The True Curves

The true (linear) curves:

$$\textbf{Demand:} \quad Q = 200 - 20P + u_D$$

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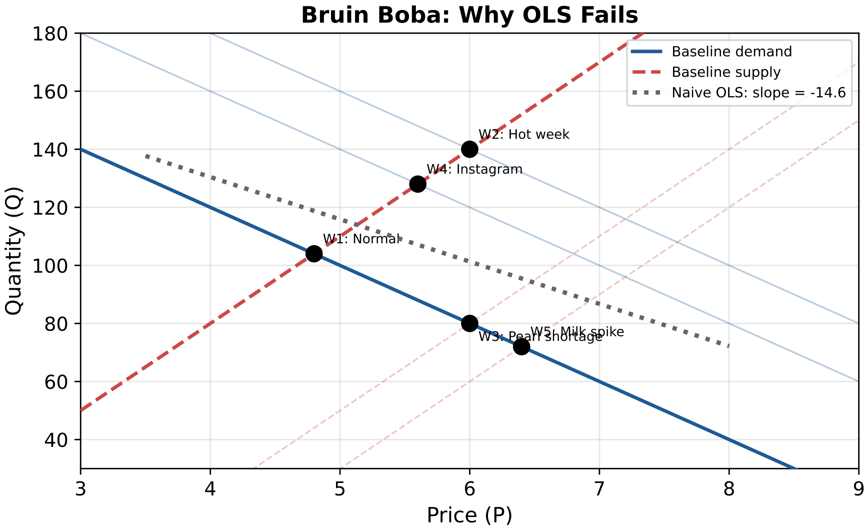
Equilibrium: Set demand = supply:

$$200 - 20P + u_D = -40 + 30P + v_S$$

$$P^* = \frac{240 + u_D - v_S}{50}, \qquad Q^* = 200 - 20P^* + u_D$$

Visualizing the Problem



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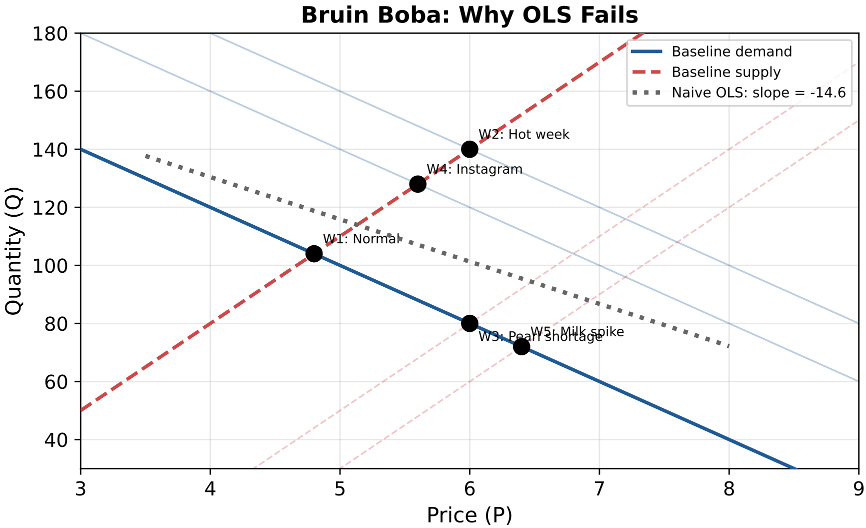

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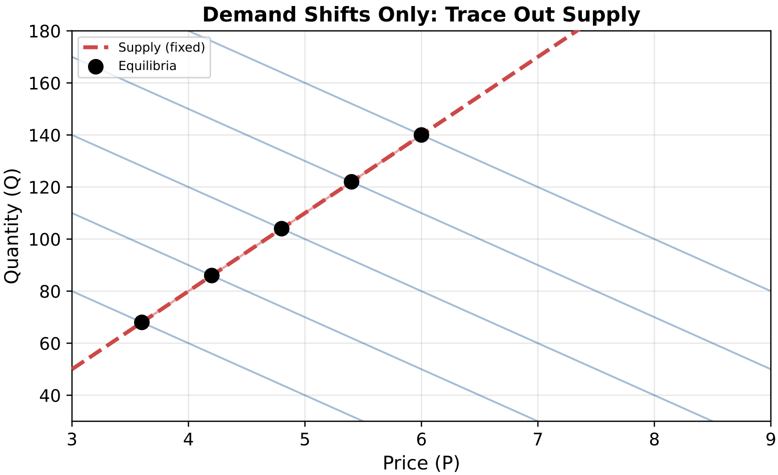

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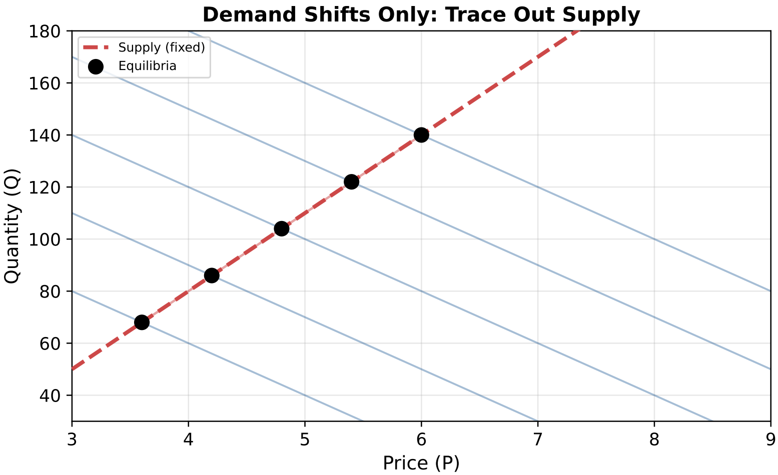

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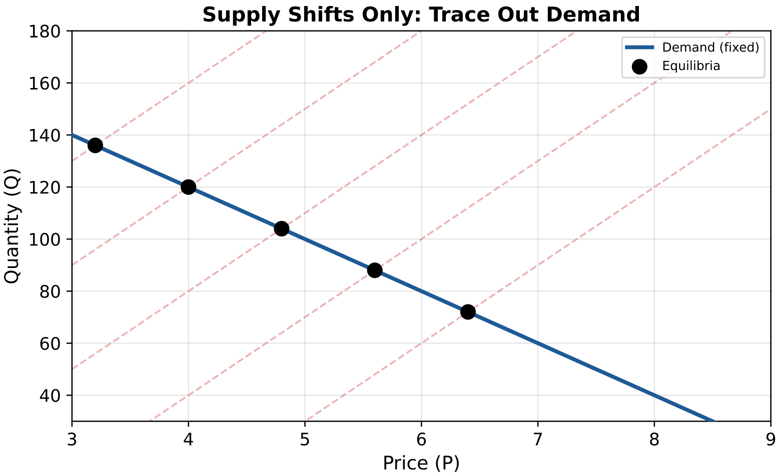

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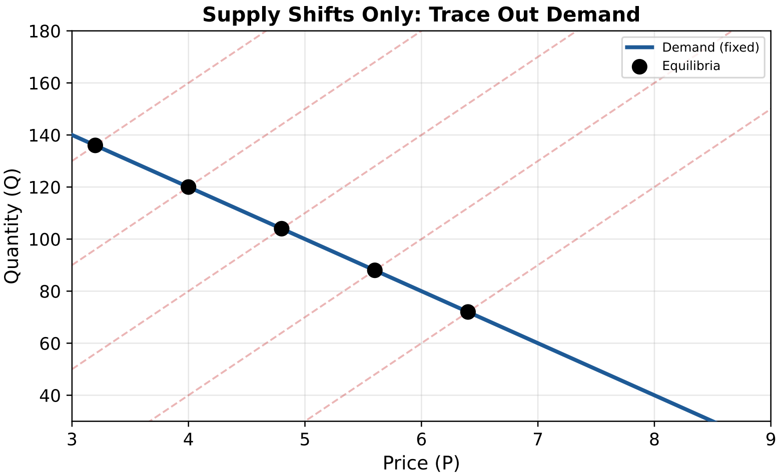

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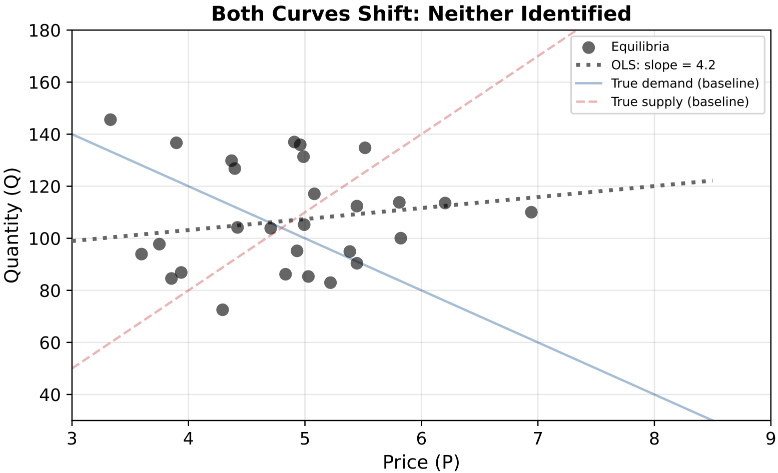

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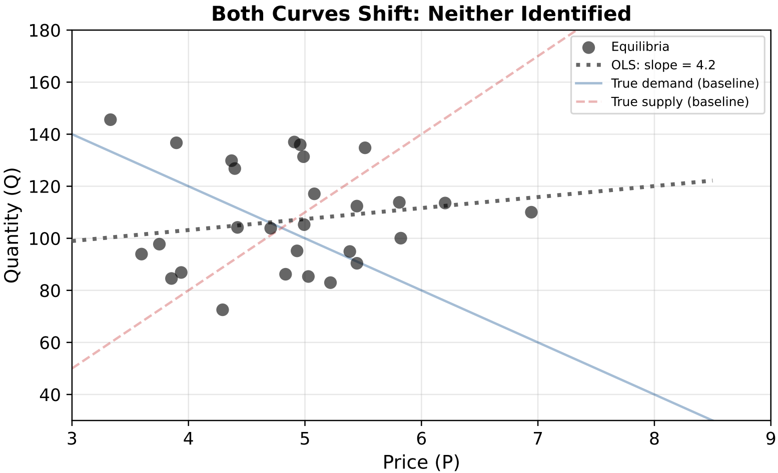

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Truffle market:

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where:

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$\Longrightarrow$ In the demand equation, P is correlated with e^d because supply shocks affect P through equilibrium. OLS on structural equations is **biased and inconsistent**.

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Derivation: Set demand = supply and solve for P:

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$\Longrightarrow$ OLS works on reduced-form equations because the RHS contains only exogenous variables. These are exactly the **first-stage regressions** in 2SLS.

Structural vs Reduced Form: Summary

Structural Form

Demand: $Q = \alpha_1 + \alpha_2 P + \alpha_3 PS + \alpha_4 DI + e^d$

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Solve for equilibrium →

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The reduced-form π coefficients are combinations of structural α's and β's. Recovering the structural parameters requires identification.

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Exclusions	vs $M-1$	Status	Meaning
$< M-1$	too few	Not identified	Cannot estimate
$= M-1$	exact	Just identified	Exactly enough instruments
$> M-1$	more than enough	Overidentified	Surplus instruments (testable)

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$\implies$ Both equations are identified. The supply equation has a surplus instrument we can test.

Identification: Visual Summary

Identified: Truffle Market

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1 exclusion
≥ M-1=1 ✓

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Not Identified: Same Variables

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If both equations contain the same variables, there are no excluded instruments and neither equation is identified.

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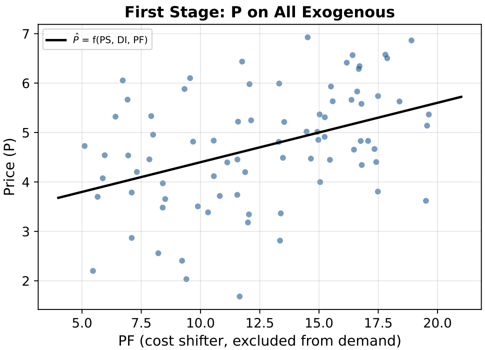
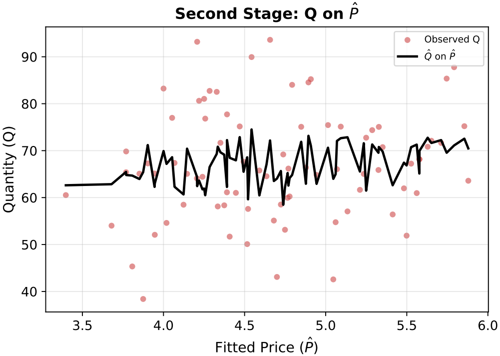
$\Longrightarrow$ The first stage purges P of its correlation with e^d. The instrument for P in the demand equation is PF (the variable excluded from demand).

2SLS: The Two Stages Visually





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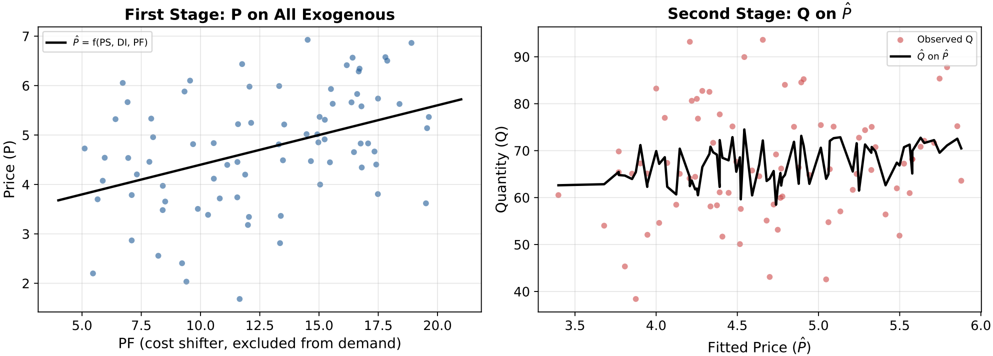

Left: First stage projects P onto exogenous variables. **Right:** Second stage uses $\hat{P}$ (purged of endogeneity) to estimate the structural equation.

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Same first stage, different second stage. Each structural equation uses different excluded variables as instruments.

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$\Longrightarrow$ The order condition is *necessary* but not *sufficient*. Instrument strength is equally important.

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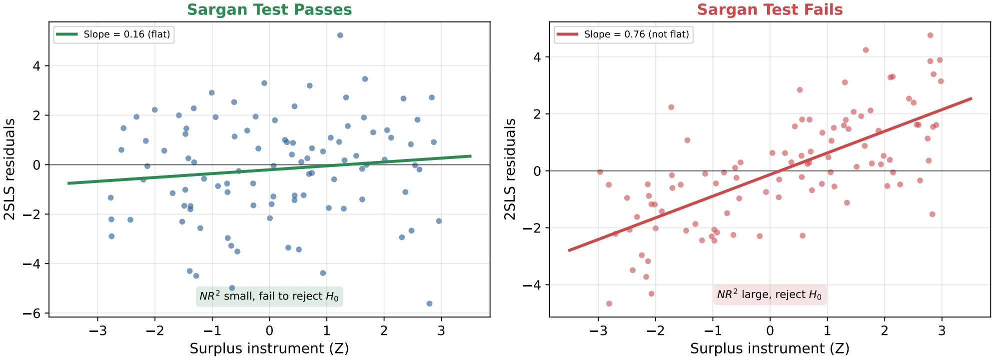
$\implies$ If $NR^2 > \chi^2_{L-B,\alpha}$, reject. The surplus instruments appear to be correlated with the error, suggesting misspecification.

Sargan Test: Visual Intuition



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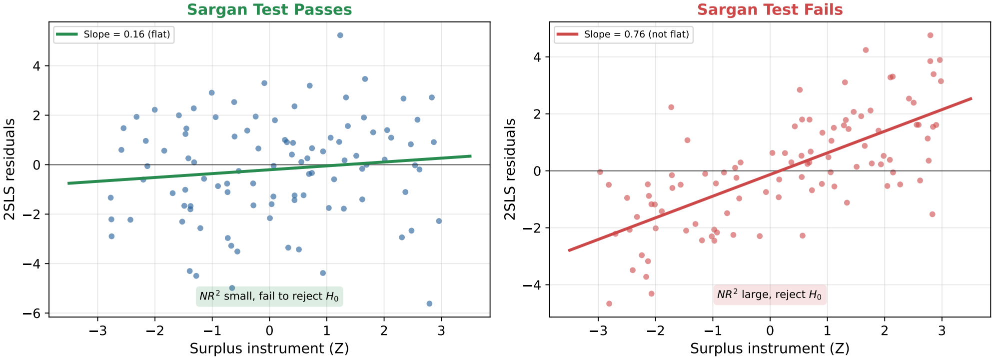

Left: Residuals show no pattern with the surplus instrument $\Longrightarrow$ valid. **Right:** Clear correlation $\Longrightarrow$ instrument is endogenous.

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Limitation: The Sargan test assumes at least one instrument is valid. If *all* instruments are bad, the test has no power to detect it.

Outline

Truffle Market: Setup

Structural model:

$$\text{Demand:} \quad Q = \alpha_1 + \alpha_2 P + \alpha_3 \text{PS} + \alpha_4 \text{DI} + e^d$$

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(b) Identification check:

- Demand: excludes PF (1 exclusion $\geq M - 1 = 1$) $\implies$ just identified
- Supply: excludes PS, DI (2 exclusions $\geq M - 1 = 1$) $\implies$ overidentified

Truffle Market: OLS vs 2SLS

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$\Longrightarrow$ Trust the 2SLS estimate (-0.53). Since PF shifts supply but not demand, it isolates movement along the demand curve.

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$\implies$ The instruments appear valid. No evidence of misspecification in the supply equation.

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4. **2SLS:** First stage regresses the endogenous variable on all exogenous variables. Second stage replaces the endogenous variable with its fitted values.
5. **Sargan test:** When overidentified, regress 2SLS residuals on all exogenous variables. $NR^2 \sim \chi^2_{L-B}$ tests instrument validity.

Thank you!

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