

Endogenous Regressors

Jake Anderson

May 16, 2026

The OLS Assumptions Under Random Sampling

Assumption	Description
RS1	The model is linear: $y_i = \beta_1 + \beta_2 x_i + e_i$
RS2	The data (y_i, x_i) are i.i.d.
RS3	Exogeneity: $E(e_i x_i) = 0$
RS4	Homoskedasticity: $\text{Var}(e_i x_i) = \sigma^2$
RS5	Rank condition: x_i takes at least two values
RS6	Normality: $e_i \sim N(0, \sigma^2)$

The OLS Assumptions Under Random Sampling

Assumption	Description
RS1	The model is linear: $y_i = \beta_1 + \beta_2 x_i + e_i$
RS2	The data (y_i, x_i) are i.i.d.
RS3	Exogeneity: $E(e_i x_i) = 0$
RS4	Homoskedasticity: $\text{Var}(e_i x_i) = \sigma^2$
RS5	Rank condition: x_i takes at least two values
RS6	Normality: $e_i \sim N(0, \sigma^2)$

We have relaxed RS4 (heteroskedasticity), RS6 (non-normality), and RS2 (time series).

The OLS Assumptions Under Random Sampling

Assumption	Description
RS1	The model is linear: $y_i = \beta_1 + \beta_2 x_i + e_i$
RS2	The data (y_i, x_i) are i.i.d.
RS3	Exogeneity: $E(e_i x_i) = 0$
RS4	Homoskedasticity: $\text{Var}(e_i x_i) = \sigma^2$
RS5	Rank condition: x_i takes at least two values
RS6	Normality: $e_i \sim N(0, \sigma^2)$

We have relaxed RS4 (heteroskedasticity), RS6 (non-normality), and RS2 (time series).

⇒ Now we tackle **RS3**: what happens when $E(e_i | x_i) \neq 0$?

The OLS Assumptions Under Random Sampling

Assumption	Description
RS1	The model is linear: $y_i = \beta_1 + \beta_2 x_i + e_i$
RS2	The data (y_i, x_i) are i.i.d.
RS3	Exogeneity: $E(e_i x_i) = 0$
RS4	Homoskedasticity: $\text{Var}(e_i x_i) = \sigma^2$
RS5	Rank condition: x_i takes at least two values
RS6	Normality: $e_i \sim N(0, \sigma^2)$

We have relaxed RS4 (heteroskedasticity), RS6 (non-normality), and RS2 (time series).

⇒ Now we tackle **RS3**: what happens when $E(e_i | x_i) \neq 0$?

Three sources of endogeneity:

- 1 Omitted variable bias
- 2 Measurement error
- 3 Simultaneity

Motivation: Test Scores

What predicts test scores? Two plausible factors:

- X_1 = prior grade (foundation from last term)
- X_2 = study time in current course

Motivation: Test Scores

What predicts test scores? Two plausible factors:

- X_1 = prior grade (foundation from last term)
- X_2 = study time in current course

The true model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$$

Motivation: Test Scores

What predicts test scores? Two plausible factors:

- X_1 = prior grade (foundation from last term)
- X_2 = study time in current course

The true model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$$

Three researchers run different regressions:

- 1 “The past is irrelevant!” $\implies Y = \beta_0 + \beta_1 X_1 + e_1$
- 2 “Effort is all that counts!” $\implies Y = \beta_0 + \beta_2 X_2 + e_2$
- 3 “Both matter!” $\implies Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$

Motivation: Test Scores

What predicts test scores? Two plausible factors:

- X_1 = prior grade (foundation from last term)
- X_2 = study time in current course

The true model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$$

Three researchers run different regressions:

- 1 “The past is irrelevant!” $\implies Y = \beta_0 + \beta_1 X_1 + e_1$
- 2 “Effort is all that counts!” $\implies Y = \beta_0 + \beta_2 X_2 + e_2$
- 3 “Both matter!” $\implies Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$

Question: Will $\hat{\beta}_1$ be the same in Model 1 and Model 3?

A Counterexample: The Room Game

Imagine a game:

- You walk down a hallway with rooms on each side
- Each room you enter gives you 1 point
- Rooms are independent of each other
- Each player randomly decides which rooms to enter (coin flip)

A Counterexample: The Room Game

Imagine a game:

- You walk down a hallway with rooms on each side
- Each room you enter gives you 1 point
- Rooms are independent of each other
- Each player randomly decides which rooms to enter (coin flip)

Model: $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$, where $\beta_1 = \beta_2 = 1$.

A Counterexample: The Room Game

Imagine a game:

- You walk down a hallway with rooms on each side
- Each room you enter gives you 1 point
- Rooms are independent of each other
- Each player randomly decides which rooms to enter (coin flip)

Model: $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$, where $\beta_1 = \beta_2 = 1$.

Now run the short regression: $Y = \beta_0 + \beta_1 x_1 + e$

A Counterexample: The Room Game

Imagine a game:

- You walk down a hallway with rooms on each side
- Each room you enter gives you 1 point
- Rooms are independent of each other
- Each player randomly decides which rooms to enter (coin flip)

Model: $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$, where $\beta_1 = \beta_2 = 1$.

Now run the short regression: $Y = \beta_0 + \beta_1 x_1 + e$

$\implies \hat{\beta}_1 \approx 1$ (no bias!)

A Counterexample: The Room Game

Imagine a game:

- You walk down a hallway with rooms on each side
- Each room you enter gives you 1 point
- Rooms are independent of each other
- Each player randomly decides which rooms to enter (coin flip)

Model: $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$, where $\beta_1 = \beta_2 = 1$.

Now run the short regression: $Y = \beta_0 + \beta_1 x_1 + e$

$\implies \hat{\beta}_1 \approx 1$ (no bias!)

Why? Because $x_1 \perp x_2$ (room choices are independent). The omitted variable x_2 does not bias $\hat{\beta}_1$ when it is **uncorrelated** with x_1 .

A Counterexample: The Room Game

Imagine a game:

- You walk down a hallway with rooms on each side
- Each room you enter gives you 1 point
- Rooms are independent of each other
- Each player randomly decides which rooms to enter (coin flip)

Model: $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$, where $\beta_1 = \beta_2 = 1$.

Now run the short regression: $Y = \beta_0 + \beta_1 x_1 + e$

$\implies \hat{\beta}_1 \approx 1$ (no bias!)

Why? Because $x_1 \perp x_2$ (room choices are independent). The omitted variable x_2 does not bias $\hat{\beta}_1$ when it is **uncorrelated** with x_1 .

\implies OVB requires both (1) omitted variable affects Y , and (2) omitted variable is correlated with included X .

Short regression (omits X_2):

$$Y_i = \beta_0 + \beta_1 X_{1i} + u_i$$

The OVB Formula

Short regression (omits X_2):

$$Y_i = \beta_0 + \beta_1 X_{1i} + u_i$$

Long regression (includes X_2):

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i$$

The OVB Formula

Short regression (omits X_2):

$$Y_i = \beta_0 + \beta_1 X_{1i} + u_i$$

Long regression (includes X_2):

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i$$

Auxiliary regression (how are X_1 and X_2 related?):

$$X_{2i} = \delta_0 + \delta_1 X_{1i} + v_i$$

The OVB Formula

Short regression (omits X_2):

$$Y_i = \beta_0 + \beta_1 X_{1i} + u_i$$

Long regression (includes X_2):

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i$$

Auxiliary regression (how are X_1 and X_2 related?):

$$X_{2i} = \delta_0 + \delta_1 X_{1i} + v_i$$

The OVB formula:

$$\hat{\beta}_1^{\text{short}} = \hat{\beta}_1^{\text{long}} + \underbrace{\hat{\beta}_2}_{\text{effect of } X_2 \text{ on } Y} \times \underbrace{\hat{\delta}_1}_{\text{relationship of } X_2 \text{ to } X_1}$$

The OVB Formula

Short regression (omits X_2):

$$Y_i = \beta_0 + \beta_1 X_{1i} + u_i$$

Long regression (includes X_2):

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i$$

Auxiliary regression (how are X_1 and X_2 related?):

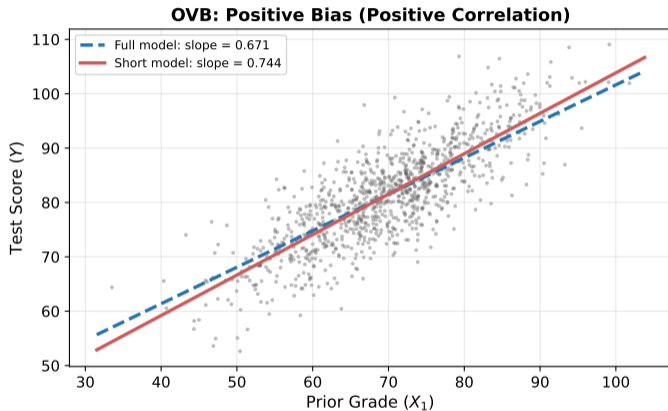
$$X_{2i} = \delta_0 + \delta_1 X_{1i} + v_i$$

The OVB formula:

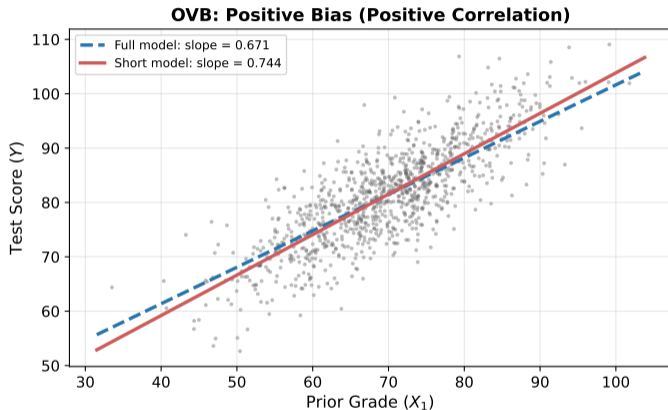
$$\hat{\beta}_1^{\text{short}} = \hat{\beta}_1^{\text{long}} + \underbrace{\hat{\beta}_2}_{\text{effect of } X_2 \text{ on } Y} \times \underbrace{\hat{\delta}_1}_{\text{relationship of } X_2 \text{ to } X_1}$$

\implies The bias is $\hat{\beta}_2 \times \hat{\delta}_1$. If either is zero, there is no bias.

Visualizing OVB: Positive Bias

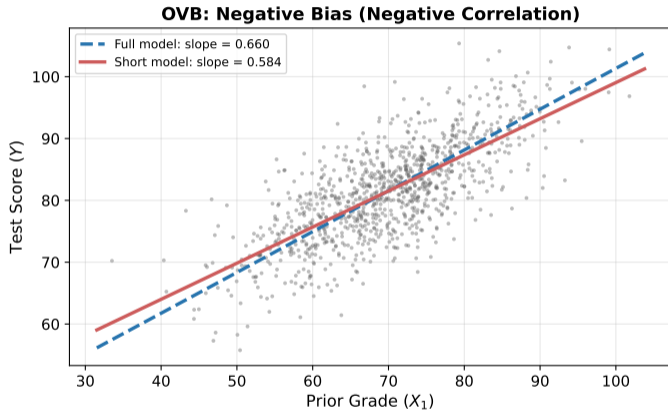


Visualizing OVB: Positive Bias

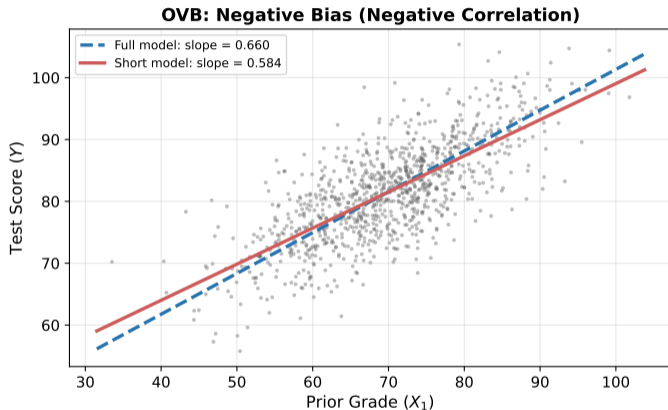


When X_1 and X_2 are **positively** correlated and $\beta_2 > 0$: the short model overestimates β_1 because it gives X_1 credit for the effect of X_2 .

Visualizing OVB: Negative Bias



Visualizing OVB: Negative Bias



When X_1 and X_2 are **negatively** correlated and $\beta_2 > 0$: the short model underestimates β_1 because the omitted variable works against X_1 .

Direction of Bias: The Sign Table

$$\hat{\beta}_1^{\text{short}} = \hat{\beta}_1^{\text{long}} + \hat{\beta}_2 \times \hat{\delta}_1$$

$\text{Corr}(X_{\text{inc}}, X_{\text{omit}}) > 0$

$\text{Corr}(X_{\text{inc}}, X_{\text{omit}}) < 0$

$\beta_{\text{omit}} > 0$

**Positive bias
(overestimate)**

**Negative bias
(underestimate)**

$\beta_{\text{omit}} < 0$

**Negative bias
(underestimate)**

**Positive bias
(overestimate)**

Direction of Bias: The Sign Table

$$\hat{\beta}_1^{\text{short}} = \hat{\beta}_1^{\text{long}} + \hat{\beta}_2 \times \hat{\delta}_1$$

	$\text{Corr}(X_{\text{inc}}, X_{\text{omit}}) > 0$	$\text{Corr}(X_{\text{inc}}, X_{\text{omit}}) < 0$
$\beta_{\text{omit}} > 0$	Positive bias (overestimate)	Negative bias (underestimate)
$\beta_{\text{omit}} < 0$	Negative bias (underestimate)	Positive bias (overestimate)

The direction of OVB depends on two signs: (1) the effect of the omitted variable on Y , and (2) the correlation between included and omitted regressors.

OVB Simulation: Test Scores

True DGP: $Y = 30 + 0.65X_1 + 0.40X_2 + \varepsilon$, with $\text{Corr}(X_1, X_2) \approx 0.4$

OVB Simulation: Test Scores

True DGP: $Y = 30 + 0.65X_1 + 0.40X_2 + \varepsilon$, with $\text{Corr}(X_1, X_2) \approx 0.4$

	Model 1 (X_1 only)	Model 2 (X_2 only)	Model 3 (Full)
$\hat{\beta}_1$ (Prior Grade)	≈ 0.73	—	≈ 0.65
$\hat{\beta}_2$ (Study Time)	—	≈ 0.55	≈ 0.40

OVB Simulation: Test Scores

True DGP: $Y = 30 + 0.65X_1 + 0.40X_2 + \varepsilon$, with $\text{Corr}(X_1, X_2) \approx 0.4$

	Model 1 (X_1 only)	Model 2 (X_2 only)	Model 3 (Full)
$\hat{\beta}_1$ (Prior Grade)	≈ 0.73	—	≈ 0.65
$\hat{\beta}_2$ (Study Time)	—	≈ 0.55	≈ 0.40

- Model 1 **overestimates** β_1 : it attributes some of study time's effect to prior grade
- Model 2 **overestimates** β_2 : it attributes some of prior grade's effect to study time
- Model 3 recovers estimates close to the true values

OVB Simulation: Test Scores

True DGP: $Y = 30 + 0.65X_1 + 0.40X_2 + \varepsilon$, with $\text{Corr}(X_1, X_2) \approx 0.4$

	Model 1 (X_1 only)	Model 2 (X_2 only)	Model 3 (Full)
$\hat{\beta}_1$ (Prior Grade)	≈ 0.73	—	≈ 0.65
$\hat{\beta}_2$ (Study Time)	—	≈ 0.55	≈ 0.40

- Model 1 **overestimates** β_1 : it attributes some of study time's effect to prior grade
- Model 2 **overestimates** β_2 : it attributes some of prior grade's effect to study time
- Model 3 recovers estimates close to the true values

⇒ Both $\beta > 0$ and $\text{Corr} > 0$, so the bias is positive (overestimation) in both short models.

Motivation: Study Time and Office Hours

Suppose test scores depend on **true study time** (x_i^*):

$$y_i = \beta_1 + \beta_2 x_i^* + v_i$$

Motivation: Study Time and Office Hours

Suppose test scores depend on **true study time** (x_i^*):

$$y_i = \beta_1 + \beta_2 x_i^* + v_i$$

But we cannot observe x_i^* directly. Instead, we use a proxy: **office hours attendance** (x_i).

Motivation: Study Time and Office Hours

Suppose test scores depend on **true study time** (x_i^*):

$$y_i = \beta_1 + \beta_2 x_i^* + v_i$$

But we cannot observe x_i^* directly. Instead, we use a proxy: **office hours attendance** (x_i).

The proxy measures true study time with error:

$$x_i = x_i^* + u_i$$

where u_i is measurement error with $E(u_i) = 0$ and $\text{Var}(u_i) = \sigma_u^2$.

Motivation: Study Time and Office Hours

Suppose test scores depend on **true study time** (x_i^*):

$$y_i = \beta_1 + \beta_2 x_i^* + v_i$$

But we cannot observe x_i^* directly. Instead, we use a proxy: **office hours attendance** (x_i).

The proxy measures true study time with error:

$$x_i = x_i^* + u_i$$

where u_i is measurement error with $E(u_i) = 0$ and $\text{Var}(u_i) = \sigma_u^2$.

Some students study a lot but never come to office hours. Others show up frequently but don't study much otherwise.

Motivation: Study Time and Office Hours

Suppose test scores depend on **true study time** (x_i^*):

$$y_i = \beta_1 + \beta_2 x_i^* + v_i$$

But we cannot observe x_i^* directly. Instead, we use a proxy: **office hours attendance** (x_i).

The proxy measures true study time with error:

$$x_i = x_i^* + u_i$$

where u_i is measurement error with $E(u_i) = 0$ and $\text{Var}(u_i) = \sigma_u^2$.

Some students study a lot but never come to office hours. Others show up frequently but don't study much otherwise.

⇒ Office hours attendance is a **noisy** version of the true variable we care about.

Why Measurement Error Causes Endogeneity

Substitute $x_i^* = x_i - u_i$ into the true model:

$$y_i = \beta_1 + \beta_2(x_i - u_i) + v_i = \beta_1 + \beta_2 x_i + \underbrace{(v_i - \beta_2 u_i)}_{e_i}$$

Why Measurement Error Causes Endogeneity

Substitute $x_i^* = x_i - u_i$ into the true model:

$$y_i = \beta_1 + \beta_2(x_i - u_i) + v_i = \beta_1 + \beta_2 x_i + \underbrace{(v_i - \beta_2 u_i)}_{e_i}$$

Now check whether the regressor x_i is correlated with the composite error e_i :

$$\text{Cov}(x_i, e_i) = \text{Cov}(x_i^* + u_i, v_i - \beta_2 u_i)$$

Why Measurement Error Causes Endogeneity

Substitute $x_i^* = x_i - u_i$ into the true model:

$$y_i = \beta_1 + \beta_2(x_i - u_i) + v_i = \beta_1 + \beta_2 x_i + \underbrace{(v_i - \beta_2 u_i)}_{e_i}$$

Now check whether the regressor x_i is correlated with the composite error e_i :

$$\begin{aligned} \text{Cov}(x_i, e_i) &= \text{Cov}(x_i^* + u_i, v_i - \beta_2 u_i) \\ &= \underbrace{\text{Cov}(x_i^*, v_i)}_{= 0} - \beta_2 \underbrace{\text{Cov}(x_i^*, u_i)}_{= 0} + \underbrace{\text{Cov}(u_i, v_i)}_{= 0} - \beta_2 \underbrace{\text{Cov}(u_i, u_i)}_{= \sigma_u^2} \end{aligned}$$

Why Measurement Error Causes Endogeneity

Substitute $x_i^* = x_i - u_i$ into the true model:

$$y_i = \beta_1 + \beta_2(x_i - u_i) + v_i = \beta_1 + \beta_2 x_i + \underbrace{(v_i - \beta_2 u_i)}_{e_i}$$

Now check whether the regressor x_i is correlated with the composite error e_i :

$$\begin{aligned}\text{Cov}(x_i, e_i) &= \text{Cov}(x_i^* + u_i, v_i - \beta_2 u_i) \\ &= \underbrace{\text{Cov}(x_i^*, v_i)}_{= 0} - \beta_2 \underbrace{\text{Cov}(x_i^*, u_i)}_{= 0} + \underbrace{\text{Cov}(u_i, v_i)}_{= 0} - \beta_2 \underbrace{\text{Cov}(u_i, u_i)}_{= \sigma_u^2} \\ &= -\beta_2 \sigma_u^2 \neq 0\end{aligned}$$

Why Measurement Error Causes Endogeneity

Substitute $x_i^* = x_i - u_i$ into the true model:

$$y_i = \beta_1 + \beta_2(x_i - u_i) + v_i = \beta_1 + \beta_2 x_i + \underbrace{(v_i - \beta_2 u_i)}_{e_i}$$

Now check whether the regressor x_i is correlated with the composite error e_i :

$$\begin{aligned}\text{Cov}(x_i, e_i) &= \text{Cov}(x_i^* + u_i, v_i - \beta_2 u_i) \\ &= \underbrace{\text{Cov}(x_i^*, v_i)}_{= 0} - \beta_2 \underbrace{\text{Cov}(x_i^*, u_i)}_{= 0} + \underbrace{\text{Cov}(u_i, v_i)}_{= 0} - \beta_2 \underbrace{\text{Cov}(u_i, u_i)}_{= \sigma_u^2} \\ &= -\beta_2 \sigma_u^2 \neq 0\end{aligned}$$

\implies If $\beta_2 > 0$, there is a **negative** correlation between x_i and e_i . OLS underestimates β_2 .

Attenuation Bias Formula

As $N \rightarrow \infty$, the OLS estimator converges to:

$$b_2 \xrightarrow{p} \beta_2 \cdot \frac{\sigma_{x^*}^2}{\underbrace{\sigma_{x^*}^2 + \sigma_u^2}_{\lambda}}$$

Attenuation Bias Formula

As $N \rightarrow \infty$, the OLS estimator converges to:

$$b_2 \xrightarrow{p} \beta_2 \cdot \underbrace{\frac{\sigma_{x^*}^2}{\sigma_{x^*}^2 + \sigma_u^2}}_{\lambda}$$

where λ is the **reliability ratio**, always between 0 and 1.

Attenuation Bias Formula

As $N \rightarrow \infty$, the OLS estimator converges to:

$$b_2 \xrightarrow{p} \beta_2 \cdot \underbrace{\frac{\sigma_{x^*}^2}{\sigma_{x^*}^2 + \sigma_u^2}}_{\lambda}$$

where λ is the **reliability ratio**, always between 0 and 1.

Two extreme cases:

- If $\sigma_u^2 = 0$ (no error): $\lambda = 1$ and $b_2 \rightarrow \beta_2$ (no bias)
- If $\sigma_u^2 \rightarrow \infty$ (pure noise): $\lambda \rightarrow 0$ and $b_2 \rightarrow 0$

Attenuation Bias Formula

As $N \rightarrow \infty$, the OLS estimator converges to:

$$b_2 \xrightarrow{p} \beta_2 \cdot \underbrace{\frac{\sigma_{x^*}^2}{\sigma_{x^*}^2 + \sigma_u^2}}_{\lambda}$$

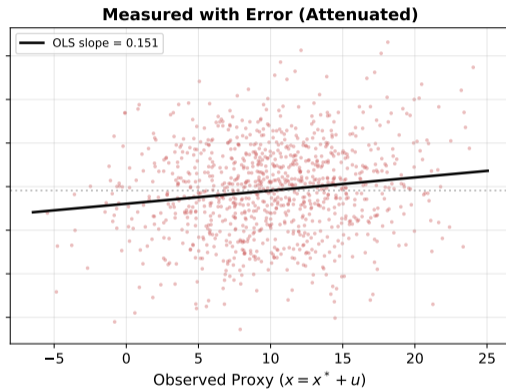
where λ is the **reliability ratio**, always between 0 and 1.

Two extreme cases:

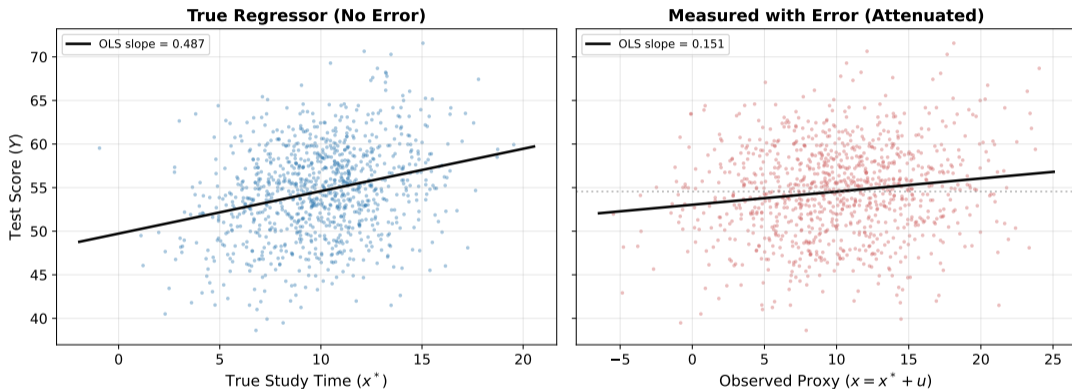
- If $\sigma_u^2 = 0$ (no error): $\lambda = 1$ and $b_2 \rightarrow \beta_2$ (no bias)
- If $\sigma_u^2 \rightarrow \infty$ (pure noise): $\lambda \rightarrow 0$ and $b_2 \rightarrow 0$

\implies Measurement error **always shrinks the coefficient toward zero**. This is called **attenuation bias**. More data does not help.

Visualizing Measurement Error

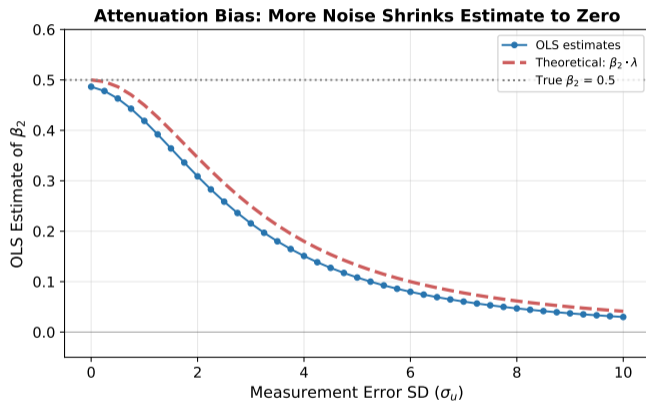


Visualizing Measurement Error

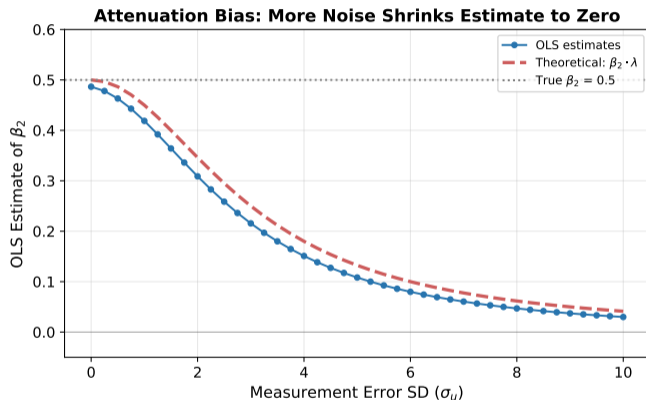


Left: using true x^* , OLS recovers the correct slope. Right: using observed x , the scatter is wider and the slope is attenuated.

Simulation: Attenuation Bias in Action

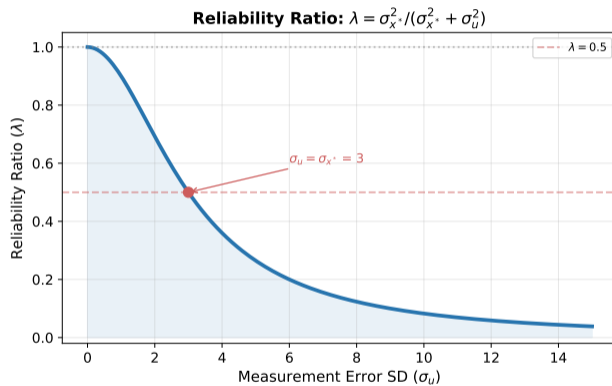


Simulation: Attenuation Bias in Action

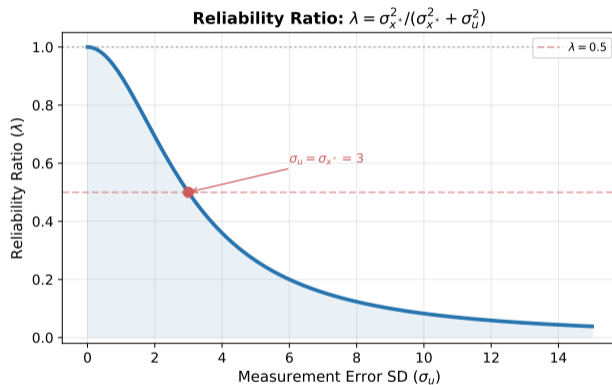


As measurement error (σ_u) increases, the OLS estimate shrinks toward zero. The theoretical curve $\beta_2 \cdot \lambda$ matches the simulated estimates.

The Reliability Ratio

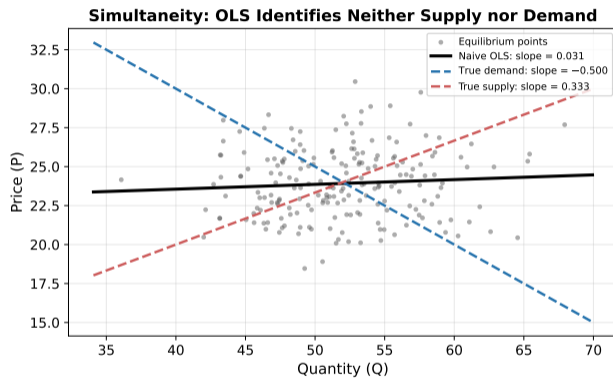


The Reliability Ratio

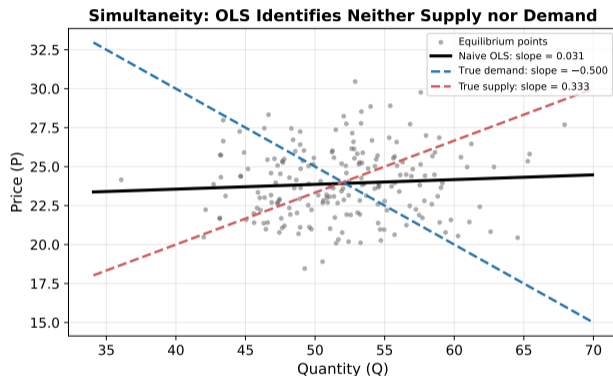


When $\sigma_u = \sigma_{x^*}$, the reliability ratio drops to 0.5: OLS captures only half of the true effect. As measurement error grows, the regressor becomes uninformative.

Simultaneity: A Brief Introduction

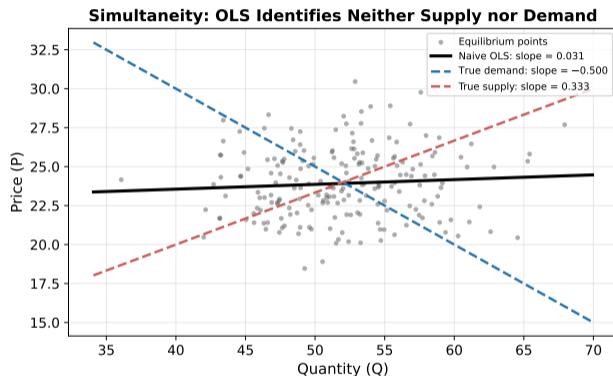


Simultaneity: A Brief Introduction



When price and quantity are determined simultaneously in equilibrium, we only observe equilibrium points. Naive OLS traces out **neither** the supply curve nor the demand curve.

Simultaneity: A Brief Introduction



When price and quantity are determined simultaneously in equilibrium, we only observe equilibrium points. Naive OLS traces out **neither** the supply curve nor the demand curve.

⇒ Covered in detail in the **Simultaneous Equations** chapter. Solution: instrumental variables.

Lagged Dependent Variable Models

A common dynamic model:

$$y_t = \beta_1 + \beta_2 y_{t-1} + \beta_3 x_t + e_t$$

Lagged Dependent Variable Models

A common dynamic model:

$$y_t = \beta_1 + \beta_2 y_{t-1} + \beta_3 x_t + e_t$$

y_{t-1} is a **random regressor**. Whether OLS works depends on the errors:

Lagged Dependent Variable Models

A common dynamic model:

$$y_t = \beta_1 + \beta_2 y_{t-1} + \beta_3 x_t + e_t$$

y_{t-1} is a **random regressor**. Whether OLS works depends on the errors:

Case 1: i.i.d. errors (e_t uncorrelated across time)

y_{t-1} was determined before e_t is realized $\implies \text{Cov}(y_{t-1}, e_t) = 0 \implies$ OLS is consistent.

Lagged Dependent Variable Models

A common dynamic model:

$$y_t = \beta_1 + \beta_2 y_{t-1} + \beta_3 x_t + e_t$$

y_{t-1} is a **random regressor**. Whether OLS works depends on the errors:

Case 1: i.i.d. errors (e_t uncorrelated across time)

y_{t-1} was determined before e_t is realized $\implies \text{Cov}(y_{t-1}, e_t) = 0 \implies$ OLS is consistent.

Case 2: AR(1) errors ($e_t = \rho e_{t-1} + v_t$)

- 1 y_{t-1} depends on e_{t-1} (from the equation at time $t-1$)
- 2 e_t depends on e_{t-1} (from the AR(1) structure)
- 3 Therefore $\text{Cov}(y_{t-1}, e_t) \neq 0$ when $\rho \neq 0$

Lagged Dependent Variable Models

A common dynamic model:

$$y_t = \beta_1 + \beta_2 y_{t-1} + \beta_3 x_t + e_t$$

y_{t-1} is a **random regressor**. Whether OLS works depends on the errors:

Case 1: i.i.d. errors (e_t uncorrelated across time)

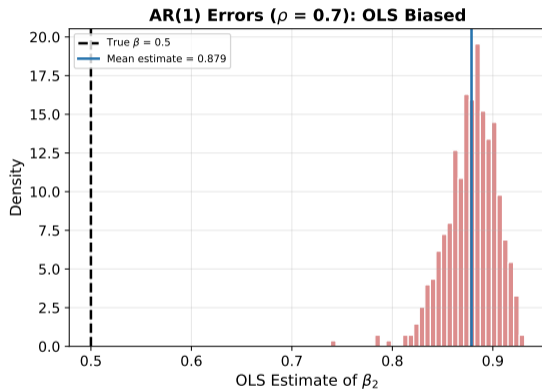
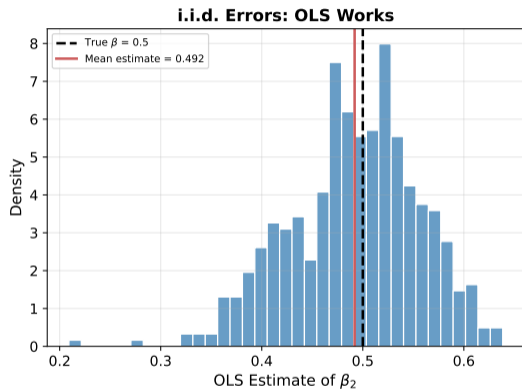
y_{t-1} was determined before e_t is realized $\implies \text{Cov}(y_{t-1}, e_t) = 0 \implies$ OLS is consistent.

Case 2: AR(1) errors ($e_t = \rho e_{t-1} + v_t$)

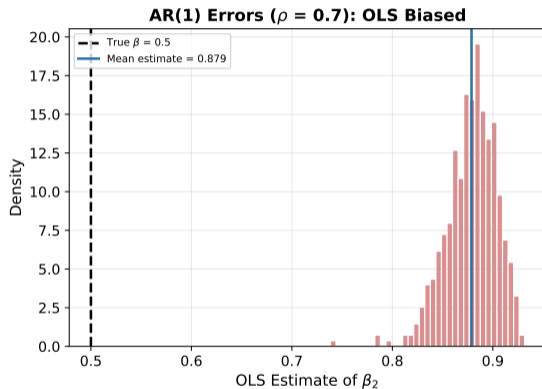
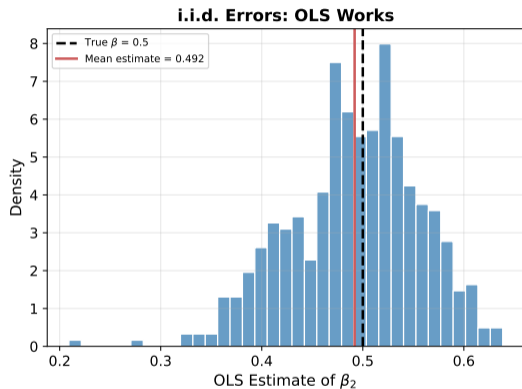
- 1 y_{t-1} depends on e_{t-1} (from the equation at time $t-1$)
- 2 e_t depends on e_{t-1} (from the AR(1) structure)
- 3 Therefore $\text{Cov}(y_{t-1}, e_t) \neq 0$ when $\rho \neq 0$

\implies Serial correlation in errors makes y_{t-1} endogenous.

Simulation: Lagged DV with Serial Correlation



Simulation: Lagged DV with Serial Correlation



Left: with i.i.d. errors, OLS is centered on the true β . Right: with AR(1) errors ($\rho = 0.7$), OLS is **biased upward**. OLS attributes error persistence to β_2 .

Problem: The standard Durbin-Watson test is invalid when y_{t-1} is a regressor.

Testing for Serial Correlation

Problem: The standard Durbin-Watson test is invalid when y_{t-1} is a regressor.

Solution: Use the **Breusch-Godfrey** test instead.

Testing for Serial Correlation

Problem: The standard Durbin-Watson test is invalid when y_{t-1} is a regressor.

Solution: Use the **Breusch-Godfrey** test instead.

Procedure:

- 1 Estimate the model and obtain residuals \hat{e}_t
- 2 Regress \hat{e}_t on all original regressors *plus* \hat{e}_{t-1} (and possibly more lags)
- 3 Test whether the coefficients on lagged residuals are jointly zero (χ^2 or F test)

Testing for Serial Correlation

Problem: The standard Durbin-Watson test is invalid when y_{t-1} is a regressor.

Solution: Use the **Breusch-Godfrey** test instead.

Procedure:

- 1 Estimate the model and obtain residuals \hat{e}_t
- 2 Regress \hat{e}_t on all original regressors *plus* \hat{e}_{t-1} (and possibly more lags)
- 3 Test whether the coefficients on lagged residuals are jointly zero (χ^2 or F test)

If the test rejects:

- The lagged dependent variable is endogenous
- OLS is inconsistent
- \implies Consider instrumental variables or other methods

Three Sources of Endogeneity

Source	Problem	Bias Direction
Omitted variables	Correlated omitted factor	Depends on signs
Measurement error	Noisy proxy for true X	Toward zero (attenuation)
Simultaneity	X and Y jointly determined	Ambiguous
Lagged DV + AR(1)	y_{t-1} correlated with e_t	Upward (when $\rho > 0$)

Three Sources of Endogeneity

Source	Problem	Bias Direction
Omitted variables	Correlated omitted factor	Depends on signs
Measurement error	Noisy proxy for true X	Toward zero (attenuation)
Simultaneity	X and Y jointly determined	Ambiguous
Lagged DV + AR(1)	y_{t-1} correlated with e_t	Upward (when $\rho > 0$)

In every case, $\text{Cov}(x_i, e_i) \neq 0$, violating RS3 (exogeneity).

Three Sources of Endogeneity

Source	Problem	Bias Direction
Omitted variables	Correlated omitted factor	Depends on signs
Measurement error	Noisy proxy for true X	Toward zero (attenuation)
Simultaneity	X and Y jointly determined	Ambiguous
Lagged DV + AR(1)	y_{t-1} correlated with e_t	Upward (when $\rho > 0$)

In every case, $\text{Cov}(x_i, e_i) \neq 0$, violating RS3 (exogeneity).

\implies More data does not fix any of these problems. The bias persists even as $N \rightarrow \infty$.

Three Sources of Endogeneity

Source	Problem	Bias Direction
Omitted variables	Correlated omitted factor	Depends on signs
Measurement error	Noisy proxy for true X	Toward zero (attenuation)
Simultaneity	X and Y jointly determined	Ambiguous
Lagged DV + AR(1)	y_{t-1} correlated with e_t	Upward (when $\rho > 0$)

In every case, $\text{Cov}(x_i, e_i) \neq 0$, violating RS3 (exogeneity).

\implies More data does not fix any of these problems. The bias persists even as $N \rightarrow \infty$.

What can we do?

- **Omitted variables:** include the variable, or use instrumental variables / fixed effects
- **Measurement error:** find a better measure, or use IV
- **Simultaneity:** use simultaneous equations methods (2SLS)
- **Lagged DV:** test with Breusch-Godfrey, use IV if errors are autocorrelated

Thank you!
jakeanderson@g.ucla.edu