

Simultaneous Equations

The Chicken and the Egg

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May 16, 2026

*“My belt holds my pants up, but the belt loops hold my belt up.
I don't really know what's happening down there.
Who is the real hero?”*

— Mitch Hedberg

Outline

- 1 Motivation: Why OLS Fails
- 2 Identification: Tracing Out Curves
- 3 Structural Form vs Reduced Form
- 4 The Identification Problem
- 5 2SLS for Simultaneous Systems
- 6 Sargan Overidentification Test
- 7 Worked Example: Truffle Market

The Problem: Jointly Determined Variables

In most regressions, we think of x as causing y . But sometimes causation runs **both ways simultaneously**.

Supply and demand:

- **Price** depends on how many people want boba (demand) and how many cups the shop can make (supply)
- **Quantity sold** also depends on the price
- P determines Q , and Q determines P : they settle together in **equilibrium**

When we observe (P_t, Q_t) data from a market, each data point is an equilibrium: the intersection of supply and demand.

⇒ Regressing Q on P with OLS does not recover the demand curve or the supply curve. It produces a confused mixture of both.

Bruin Boba: The Setup

Track weekly **price per cup** (P_t) and **cups sold** (Q_t) at Bruin Boba:

Week	Shock	P^*	Q^*
1	Normal	\$4.80	104
2	Hot week (demand \uparrow)	\$6.00	140
3	Pearl shortage (supply \downarrow)	\$6.00	80
4	Instagram viral (demand \uparrow)	\$5.60	128
5	Milk spike (supply \downarrow)	\$6.40	72

Notice: Weeks 2 and 3 have the **same price** (\$6.00) but very different quantities (140 vs 80). One is a demand shift, the other is a supply shift.

\implies A single OLS line through (P_t, Q_t) cannot recover demand or supply.

Bruin Boba: The True Curves

The true (linear) curves:

$$\text{Demand: } Q = 200 - 20P + u_D$$

$$\text{Supply: } Q = -40 + 30P + v_S$$

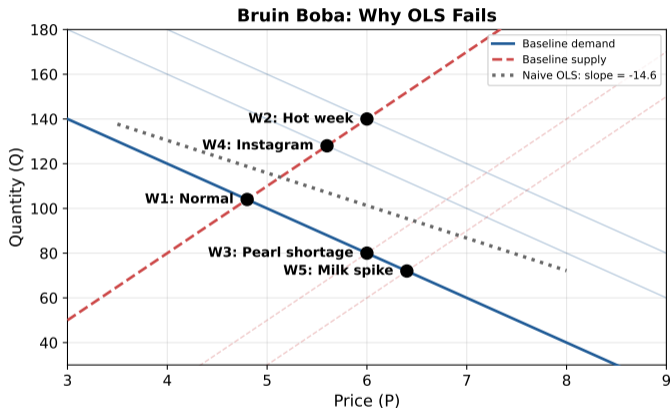
where u_D = demand shifter (weather, virality) and v_S = supply shifter (input costs).

Equilibrium: Set demand = supply:

$$200 - 20P + u_D = -40 + 30P + v_S$$

$$P^* = \frac{240 + u_D - v_S}{50}, \quad Q^* = 200 - 20P^* + u_D$$

Visualizing the Problem



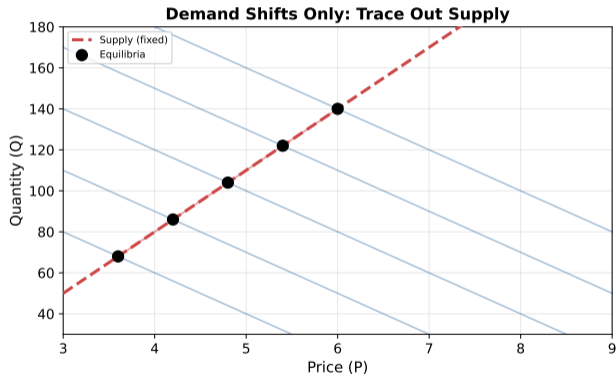
The gray dotted line is naive OLS. It does not match the demand curve (blue) or the supply curve (red). This is **simultaneity bias**.

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When Only Demand Shifts

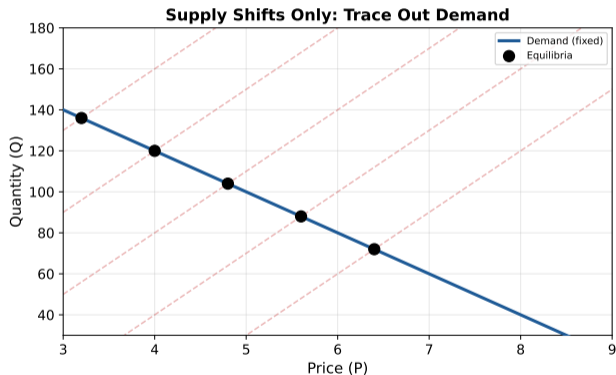
If demand shifts while supply stays fixed, equilibria move **along the supply curve**.



⇒ Demand shifters serve as instruments for identifying the supply equation.

When Only Supply Shifts

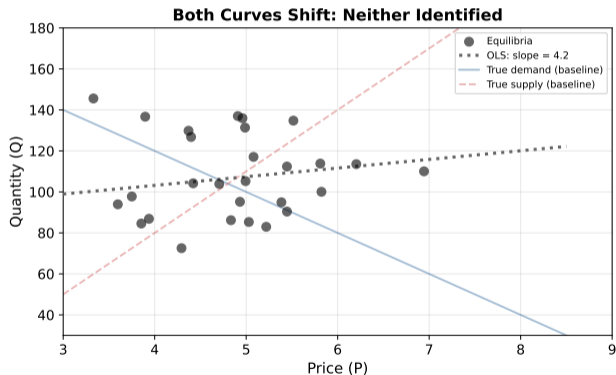
If supply shifts while demand stays fixed, equilibria move **along the demand curve**.



⇒ Supply shifters serve as instruments for identifying the demand equation.

When Both Shift: No Identification

If both curves shift simultaneously and we have no way to separate the shifts, the equilibria scatter in both directions.



⇒ Without excluded instruments, we cannot identify either curve.

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The Structural Form

The **structural form** describes behavioral relationships from economic theory.

Truffle market:

$$\text{Demand: } Q = \alpha_1 + \alpha_2 P + \alpha_3 PS + \alpha_4 DI + e^d$$

$$\text{Supply: } Q = \beta_1 + \beta_2 P + \beta_3 PF + e^s$$

where:

- P, Q = endogenous (determined *within* the system)
- PS = price of substitute, DI = disposable income (exogenous, demand side)
- PF = price of a production factor (exogenous, supply side)

\implies In the demand equation, P is correlated with e^d because supply shocks affect P through equilibrium. OLS on structural equations is **biased and inconsistent**.

The Reduced Form

The **reduced form** expresses each endogenous variable as a function of *only* exogenous variables.

Derivation: Set demand = supply and solve for P :

$$P = \underbrace{\frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}}_{\pi_{10}} + \underbrace{\frac{-\alpha_3}{\alpha_2 - \beta_2}}_{\pi_{11}} PS + \underbrace{\frac{-\alpha_4}{\alpha_2 - \beta_2}}_{\pi_{12}} DI + \underbrace{\frac{\beta_3}{\alpha_2 - \beta_2}}_{\pi_{13}} PF + v_1$$

Written compactly:

$$P = \pi_{10} + \pi_{11}PS + \pi_{12}DI + \pi_{13}PF + v_1$$

$$Q = \pi_{20} + \pi_{21}PS + \pi_{22}DI + \pi_{23}PF + v_2$$

\implies OLS works on reduced-form equations because the RHS contains only exogenous variables. These are exactly the **first-stage regressions** in 2SLS.

Structural vs Reduced Form: Summary

Structural Form

$$\text{Demand: } Q = \alpha_1 + \alpha_2 P + \alpha_3 PS + \alpha_4 DI + e^d$$

$$\text{Supply: } Q = \beta_1 + \beta_2 P + \beta_3 PF + e^s$$

Solve for
equilibrium



Reduced Form

$$P = \pi_{10} + \pi_{11} PS + \pi_{12} DI + \pi_{13} PF + v_1$$

$$Q = \pi_{20} + \pi_{21} PS + \pi_{22} DI + \pi_{23} PF + v_2$$

RHS of reduced form = only exogenous variables \Rightarrow OLS is consistent

The reduced-form π coefficients are combinations of structural α 's and β 's. Recovering the structural parameters requires identification.

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The Order Condition

In a system of M simultaneous equations, an equation is identified if it **excludes at least** $M - 1$ variables that appear elsewhere in the system.

Intuition: To trace out the demand curve, we need something that shifts *supply* but not demand. The excluded variables provide those shifts.

Exclusions	vs $M - 1$	Status	Meaning
$< M - 1$	too few	Not identified	Cannot estimate
$= M - 1$	exact	Just identified	Exactly enough instruments
$> M - 1$	more than enough	Overidentified	Surplus instruments (testable)

Checking Identification: Truffle Market

With $M = 2$ equations, we need at least $M - 1 = 1$ exclusion per equation.

Demand equation: $Q = \alpha_1 + \alpha_2 P + \alpha_3 PS + \alpha_4 DI + e^d$

- Contains: P , Q , PS , DI
- Excludes: **PF** (appears in supply, not demand)
- 1 exclusion $\geq 1 \implies$ **just identified**

Supply equation: $Q = \beta_1 + \beta_2 P + \beta_3 PF + e^s$

- Contains: P , Q , PF
- Excludes: **PS** and **DI** (appear in demand, not supply)
- 2 exclusions $\geq 1 \implies$ **overidentified**

\implies Both equations are identified. The supply equation has a surplus instrument we can test.

Identification: Visual Summary

Identified: Truffle Market

Demand equation

$$Q = \alpha_1 + \alpha_2 P + \alpha_3 PS + \alpha_4 DI + e^d$$

Contains: P, Q, PS, DI
Excludes: PF

1 exclusion
 $\geq M-1=1$ ✓

Supply equation

$$Q = \beta_1 + \beta_2 P + \beta_3 PF + e^s$$

Contains: P, Q, PF
Excludes: PS, DI

2 exclusions
 $\geq M-1=1$ ✓

Just identified / Overidentified

Not Identified: Same Variables

Demand equation

$$Q = \alpha_1 + \alpha_2 P + \alpha_3 DI + e^d$$

Contains: P, Q, DI
Excludes: nothing

0 exclusions
 $< M-1=1$ ✗

Supply equation

$$Q = \beta_1 + \beta_2 P + \beta_3 DI + e^s$$

Contains: P, Q, DI
Excludes: nothing

0 exclusions
 $< M-1=1$ ✗

Neither equation identified

If both equations contain the same variables, there are no excluded instruments and neither equation is identified.

Why an Excluded Variable Identifies the Other Curve

Every equilibrium (P, Q) lies on *both* curves at once. To learn either curve we need a variable that moves one curve while the other stays put.

Identified case. The price of a production input shifts supply but does not enter demand. As that input price changes week to week, supply slides up and down while demand stays put. The shifted supply curves cut demand at different points, so the equilibrium price-quantity pairs *trace out* the demand curve. The price of a substitute and disposable income do the same job in reverse: they shift demand only, so the equilibrium pairs trace out supply. \implies Both curves recoverable.

Not-identified case. Every observable variable enters *both* equations, so whenever something changes, both curves shift together. The new equilibrium could be anywhere. The scatter of price-quantity points doesn't trace out either curve, just the joint movement. \implies No estimator (OLS, 2SLS, anything) can separate demand from supply with this data alone.

\implies The excluded variable is what lets your data answer “which curve is which.” Without one, the question is unanswerable, not just hard.

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2SLS: Why Stage 1 Is the Reduced Form for P

We want to estimate the structural demand equation:

$$Q = \alpha_1 + \alpha_2 P + \alpha_3 PS + \alpha_4 DI + e^d.$$

OLS is biased because P is endogenous: supply shocks e^s move P , and through the simultaneous system, P ends up correlated with e^d .

Fix: replace P with a version that depends *only* on the exogenous variables of the system. Get there by setting demand = supply and solving for P :

$$\alpha_1 + \alpha_2 P + \alpha_3 PS + \alpha_4 DI + e^d = \beta_1 + \beta_2 P + \beta_3 PF + e^s.$$

Collect the P terms on the left and divide by $(\alpha_2 - \beta_2)$:

$$P = \underbrace{\frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}}_{\pi_{10}} + \underbrace{\frac{-\alpha_3}{\alpha_2 - \beta_2}}_{\pi_{11}} PS + \underbrace{\frac{-\alpha_4}{\alpha_2 - \beta_2}}_{\pi_{12}} DI + \underbrace{\frac{\beta_3}{\alpha_2 - \beta_2}}_{\pi_{13}} PF + v_1.$$

\implies This is the **reduced form for P** : P written purely as a function of exogenous variables. Because the right-hand side is exogenous, *OLS on this equation is consistent*. That regression is Stage 1.

2SLS: The Two Stages, Clearly

Stage 1: estimate the reduced form for P and form \hat{P} .

$$P = \pi_{10} + \pi_{11}PS + \pi_{12}DI + \pi_{13}PF + v_1 \xrightarrow{\text{OLS}} \hat{P}_i = \hat{\pi}_{10} + \hat{\pi}_{11}PS_i + \hat{\pi}_{12}DI_i + \hat{\pi}_{13}PF_i.$$

\hat{P} is the part of P explained by exogenous variables; the leftover v_1 carries the endogenous noise.

Stage 2: replace P with \hat{P} in the structural demand equation and run OLS.

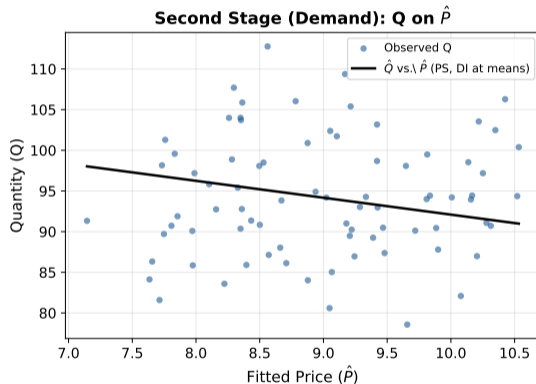
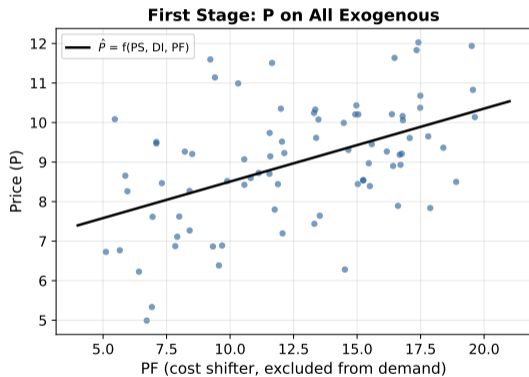
$$Q = \alpha_1 + \alpha_2\hat{P} + \alpha_3PS + \alpha_4DI + \text{residual}.$$

The coefficient on \hat{P} is the 2SLS estimate of α_2 (the demand slope).

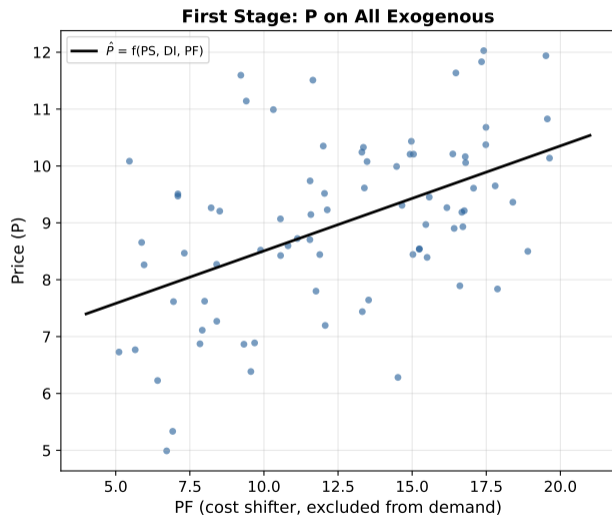
Why it works.

- \hat{P} depends only on exogenous variables, so $\text{Cov}(\hat{P}, e^d) = 0$ in the limit.
- The exclusion that makes 2SLS identify demand is the price of a production input: it appears in \hat{P} (because it shifts supply) but does *not* enter the demand equation itself \implies it's the instrument.
- Standard errors should come from a 2SLS-aware routine (e.g. `ivreg` in R), not from running two OLS regressions by hand.

2SLS: The Two Stages Visually

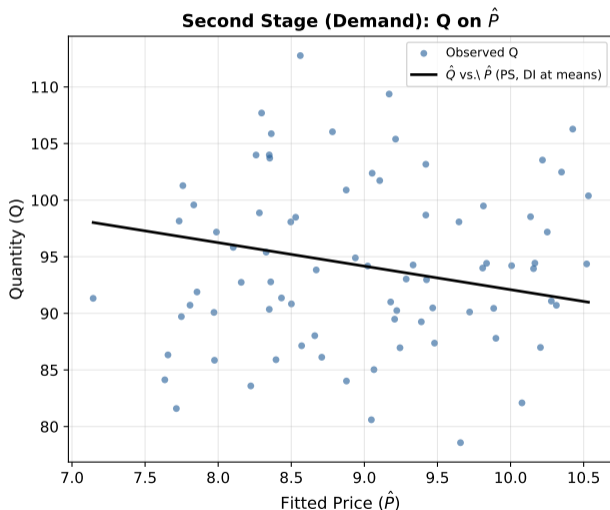


First Stage: What to Notice



- **Nonzero slope.** The fitted line clearly tilts up \implies the input price moves P . A near-flat line would mean a weak instrument and 2SLS does not work.
- **Sign matches theory.** Higher input cost \implies higher price. The supply curve shifts up, equilibrium price rises.

Second Stage: What to Notice



- **Negative slope.** That slope *is* the 2SLS estimate of α_2 , the demand response: higher price \implies lower quantity. Sign matches downward-sloping demand.
- **Why \hat{P} instead of P ?** Raw P contains supply-side shocks correlated with e^d , which biases OLS. \hat{P} keeps only the part of P driven by exogenous variables \implies the slope captures pure demand response.

2SLS for the Supply Equation

Estimating supply ($Q = \beta_1 + \beta_2 P + \beta_3 PF + e^s$):

Stage 1: Same regression as before (P on all exogenous):

$$P = \pi_{10} + \pi_{11}PS + \pi_{12}DI + \pi_{13}PF + v_1 \quad \longrightarrow \quad \hat{P}$$

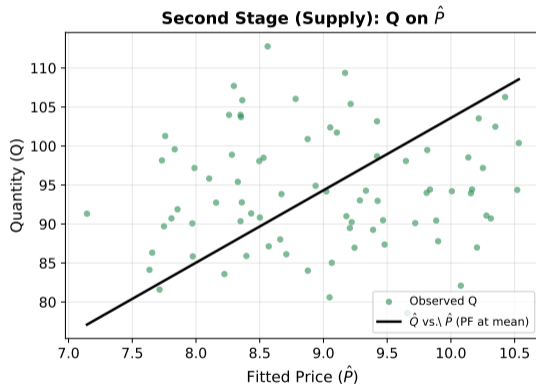
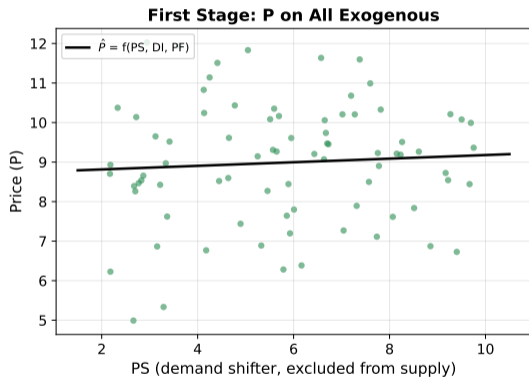
Stage 2: Replace P with \hat{P} in the supply equation:

$$Q = \beta_1 + \beta_2 \hat{P} + \beta_3 PF + \text{residual}$$

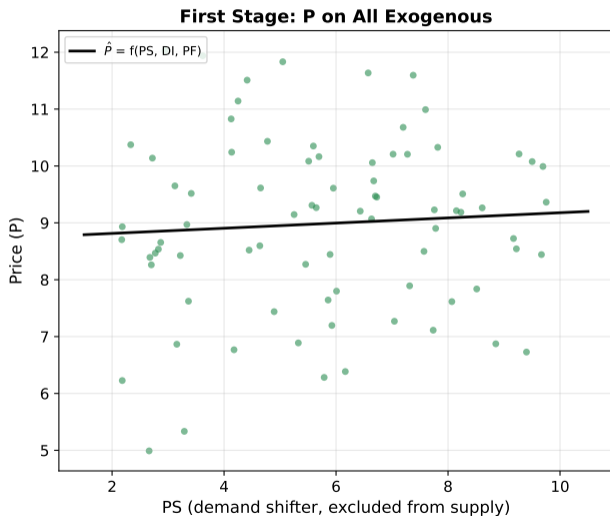
The instruments for P in the supply equation are PS and DI (the demand shifters excluded from supply).

Same first stage, different second stage. Each structural equation uses different excluded variables as instruments.

2SLS for Supply: The Two Stages Visually

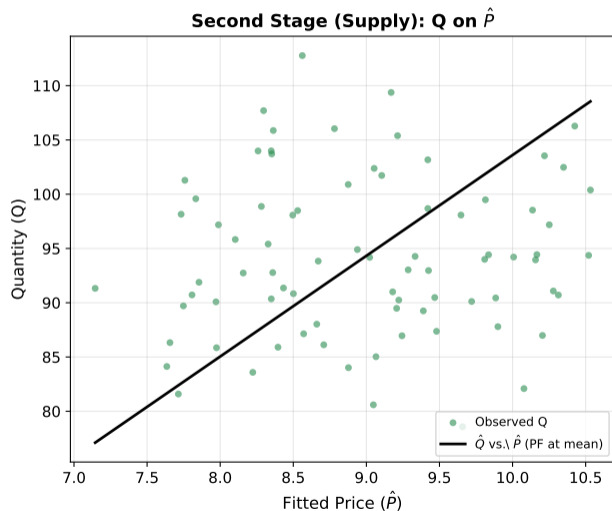


First Stage (Supply): What to Notice



- **Nonzero slope.** PS (the substitute price, excluded from supply) clearly moves $P \implies$ it is a relevant instrument for the supply equation.
- **Sign matches theory.** A higher substitute price shifts demand for our good outward, pushing equilibrium price up.

Second Stage (Supply): What to Notice



- **Positive slope.** That slope *is* the 2SLS estimate of β_2 , the supply response: higher price \implies higher quantity. Sign matches upward-sloping supply.
- **Why \hat{P} instead of P ?** Raw P contains demand-side shocks correlated with e^S . \hat{P} keeps only the part of P driven by exogenous variables \implies the slope captures pure supply response.

Same rule as single-equation IV: check that the instruments are **strong**.

- The F -statistic from the first-stage regression should exceed **10** (Staiger–Stock rule of thumb)
- Weak instruments lead to biased and imprecise 2SLS estimates

In a simultaneous system:

- The first stage regresses the endogenous variable on **all** exogenous variables in the entire system
- Excluded instruments provide the identifying variation
- If the excluded instruments are weak predictors of P , identification breaks down even though the order condition is satisfied

⇒ The order condition is *necessary* but not *sufficient*. Instrument strength is equally important.

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Testing Surplus Instruments: Sargan Test

When an equation is **overidentified** ($L > B$, where L = number of instruments, B = number of endogenous regressors), we have **surplus instruments** we can test.

Sargan test procedure:

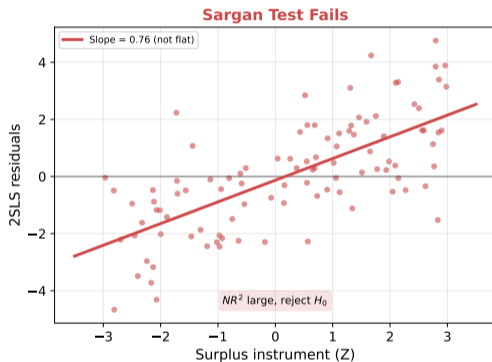
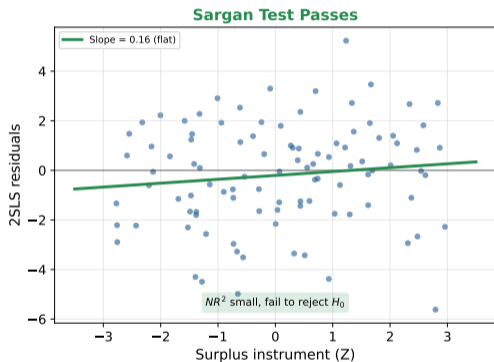
- 1 Estimate the structural equation by 2SLS. Obtain residuals \hat{e} .
- 2 Regress \hat{e} on all exogenous variables (the instruments). Get R^2 .
- 3 Test statistic: $NR^2 \sim \chi_{L-B}^2$ under H_0 .

H_0 : All surplus instruments are valid (uncorrelated with the structural error).

H_1 : At least one surplus instrument is invalid.

\implies If $NR^2 > \chi_{L-B, \alpha}^2$, reject. The surplus instruments appear to be correlated with the error, suggesting misspecification.

Sargan Test: Visual Intuition



Left: Residuals show no pattern with the surplus instrument \implies valid. **Right:** Clear correlation \implies instrument is endogenous.

Sargan Test: Truffle Supply Equation

The supply equation has $L = 2$ instruments (PS, DI) for $B = 1$ endogenous variable (P).

$\implies L - B = 1$ surplus instrument. Degrees of freedom = 1.

Example result: $NR^2 = 0.43$, critical value $\chi_{1,0.05}^2 = 3.84$.

Since $0.43 < 3.84$, we **fail to reject** H_0 .

\implies No evidence that the surplus instrument is invalid. The 2SLS estimates for the supply equation appear reliable.

Limitation: The Sargan test assumes at least one instrument is valid. If *all* instruments are bad, the test has no power to detect it.

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Structural model:

$$\text{Demand: } Q = \alpha_1 + \alpha_2 P + \alpha_3 PS + \alpha_4 DI + e^d$$

$$\text{Supply: } Q = \beta_1 + \beta_2 P + \beta_3 PF + e^s$$

(a) Endogenous vs exogenous:

- **Endogenous:** P and Q (jointly determined by the intersection)
- **Exogenous:** PS (substitute price), DI (disposable income), PF (factor cost)

(b) Identification check:

- Demand: excludes PF (1 exclusion $\geq M - 1 = 1$) \implies just identified
- Supply: excludes PS, DI (2 exclusions $\geq M - 1 = 1$) \implies overidentified

(c) A researcher naively runs OLS on the demand equation:

$$\hat{\alpha}_2^{\text{OLS}} = -0.37$$

The 2SLS estimate (using PF as instrument for P):

$$\hat{\alpha}_2^{\text{2SLS}} = -0.53$$

Why do they differ?

- OLS is biased because $\text{Cov}(P, e^d) \neq 0$
- The OLS estimate (-0.37) **understates** the true price sensitivity: simultaneity bias pushes the estimate toward zero

\implies Trust the 2SLS estimate (-0.53). Since PF shifts supply but not demand, it isolates movement along the demand curve.

Truffle Market: Overidentification Test

(d) For the supply equation, the Sargan overidentification test gives:

$$NR^2 = 0.43, \quad \chi_{1, 0.05}^2 = 3.84$$

H_0 : All surplus instruments are valid (uncorrelated with e^s).

Since $0.43 < 3.84$: **fail to reject** H_0 .

The supply equation has $L = 2$ instruments (PS, DI) for $B = 1$ endogenous variable (P), so $L - B = 1$ surplus instrument can be tested.

\implies The instruments appear valid. No evidence of misspecification in the supply equation.

- 1 **Simultaneity bias:** When P and Q are jointly determined, OLS on structural equations is biased and inconsistent
- 2 **Identification:** To estimate a structural equation, we need variables excluded from that equation but present elsewhere (the order condition: $\geq M - 1$ exclusions)
- 3 **Reduced form:** Expresses endogenous variables as functions of only exogenous variables. OLS is consistent on reduced-form equations.
- 4 **2SLS:** First stage regresses the endogenous variable on all exogenous variables. Second stage replaces the endogenous variable with its fitted values.
- 5 **Sargan test:** When overidentified, regress 2SLS residuals on all exogenous variables.
 $NR^2 \sim \chi^2_{L-B}$ tests instrument validity.

Thank you!
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