Introduction to Fixed Effects

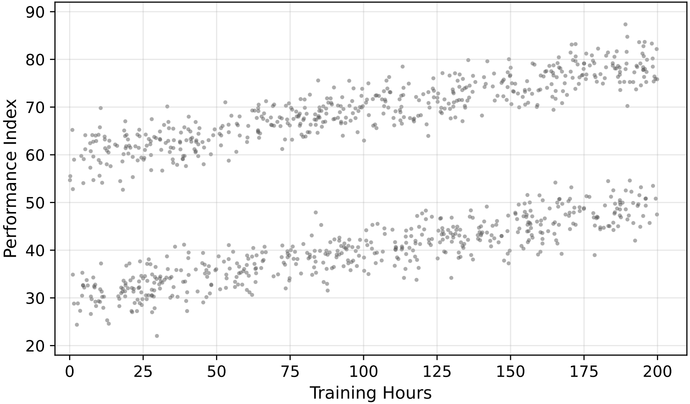
Why One Regression Line Isn't Enough

Jake Anderson

May 16, 2026

The Data

A coach tracks **training hours** vs. **performance** for their trainees.



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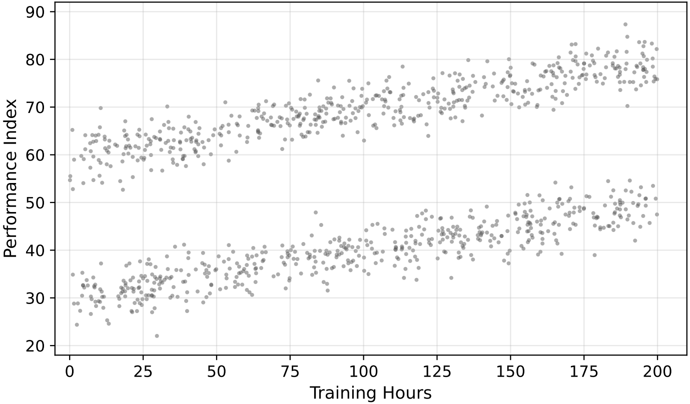

How could this data be generated?

The Setup

It turns out there are **two teams**: Varsity and Junior Varsity (JV). Same training program, different baseline ability.

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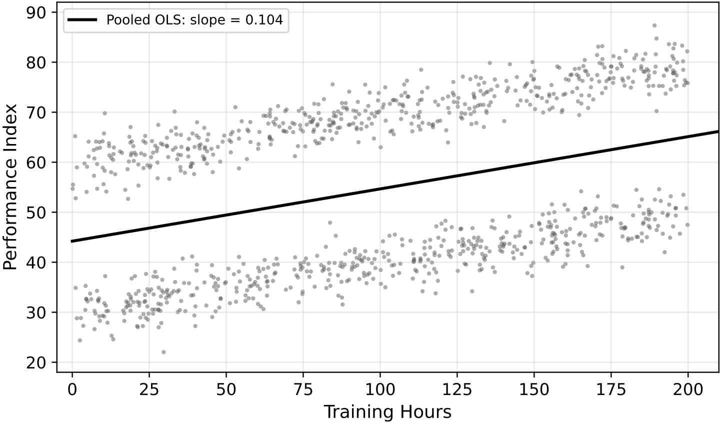
Question: What goes wrong when we force one intercept on data with two?

What Happens If We Run Naive OLS?



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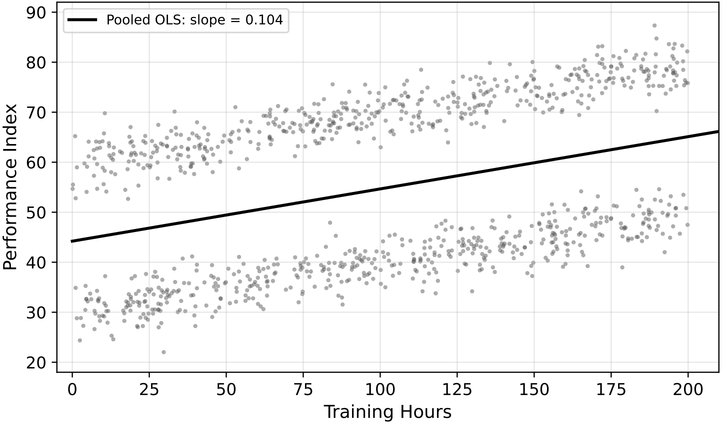

Pooled OLS: slope ≈ 0.10 — looks correct! But this only works because the sample is balanced (50/50) and x is identically distributed across groups. What if that changes?

First Check: Are Groups Sampled Differently?

In practice, the distribution of x often differs across groups.

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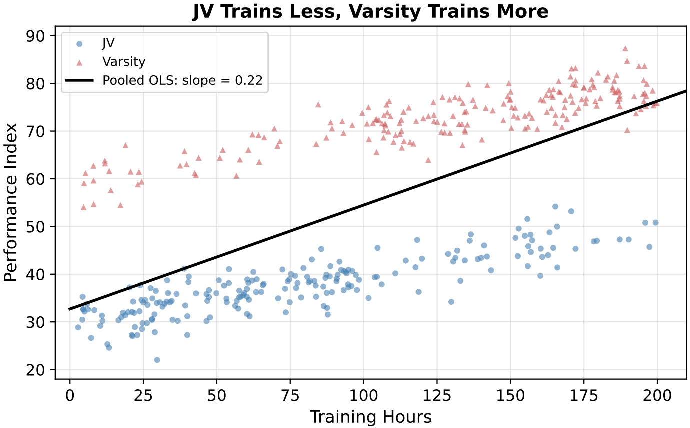
If so, pooled OLS conflates the group effect (α_j) with the treatment effect (β).

This is **omitted variable bias**.

Scenario: Varsity Trains More



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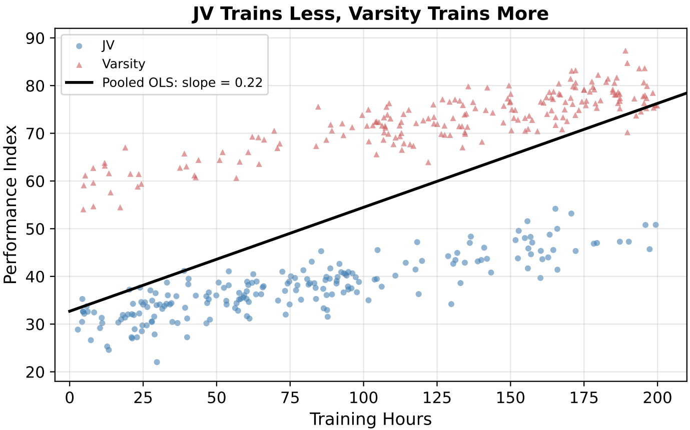
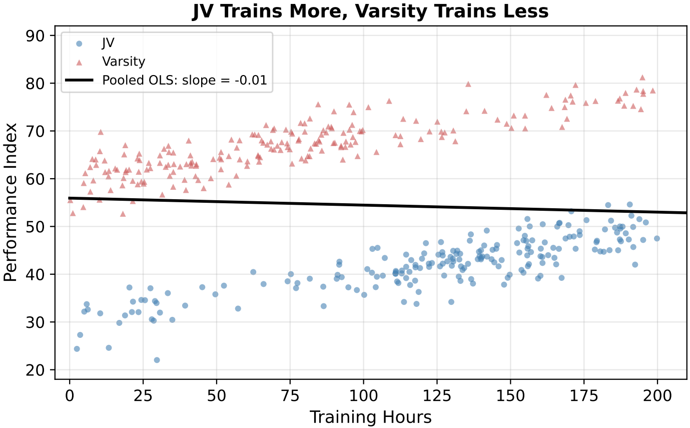


OLS slope = **0.22** (true = 0.10). Bias is **positive**: OLS attributes Varsity's higher baseline to their greater training hours.

Scenario: JV Trains More (Catching Up)



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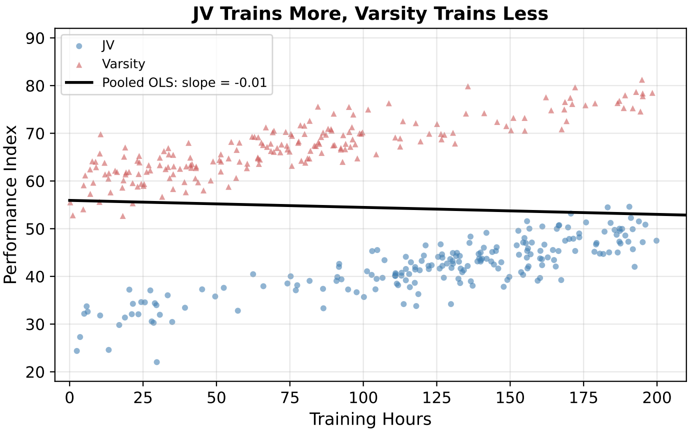

OLS slope $= -\mathbf{0.02}$ (true $= 0.10$). Bias is **negative**: OLS thinks training has nearly *zero effect* because the high-training group has lower baseline ability.

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OVB formula: (effect of X_2 on Y) $\times$ (relationship of X_2 to X_1)

$$\hat{\beta}_1^{\text{short}} = \hat{\beta}_1^{\text{long}} + \hat{\beta}_2 \times \hat{\delta}_1$$

OVB Applied to Our Example

$$\hat{\beta}_1^{\text{short}} = \underbrace{\hat{\beta}_1^{\text{long}}}_{=0.10} + \underbrace{\hat{\beta}_2}_{\substack{\text{effect of group} \\ \text{on Performance}}} \times \underbrace{\hat{\delta}_1}_{\substack{\text{relationship of} \\ \text{group to Hours}}}$$

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Varsity trains more	> 0	$+$	0.22
JV trains more	< 0	$-$	-0.02
Equal training	≈ 0	≈ 0	0.10

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Same data, same true effect. The OLS estimate swings from -0.02 to $+0.22$ just by changing which group trains more.

What About Class Imbalance?

Suppose x is distributed the same across groups ($\delta \approx 0$), but the **sample composition** is unbalanced.

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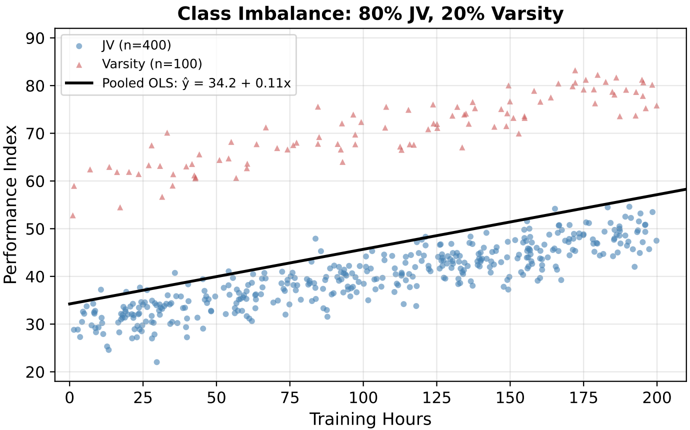
The OLS intercept is a **weighted average** of the group intercepts:

$$
\begin{aligned}
\hat{\beta}_0 &= \frac{\sum_{i=1}^{n} \mathbb{1}\{i \in \text{JV}\}}{n} \cdot \beta_{0,\text{JV}} + \frac{\sum_{i=1}^{n} \mathbb{1}\{i \in \text{Var}\}}{n} \cdot \beta_{0,\text{Var}} \\
&= \frac{n_{\text{JV}}}{n} \cdot \beta_{0,\text{JV}} + \frac{n_{\text{Var}}}{n} \cdot \beta_{0,\text{Var}} \\
&= (\text{Share JV}) \cdot \beta_{0,\text{JV}} + (\text{Share Var}) \cdot \beta_{0,\text{Var}}
\end{aligned}
$$

80% JV, 20% Varsity



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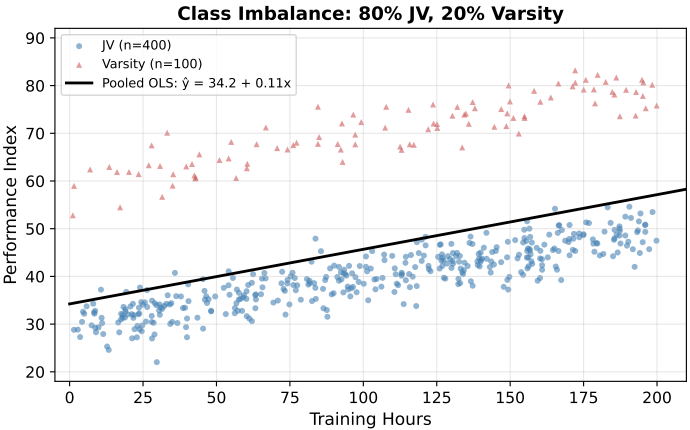

Slope ≈ correct. Intercept = **34** — pulled toward JV's true intercept (30). Predictions are wrong for most Varsity players.

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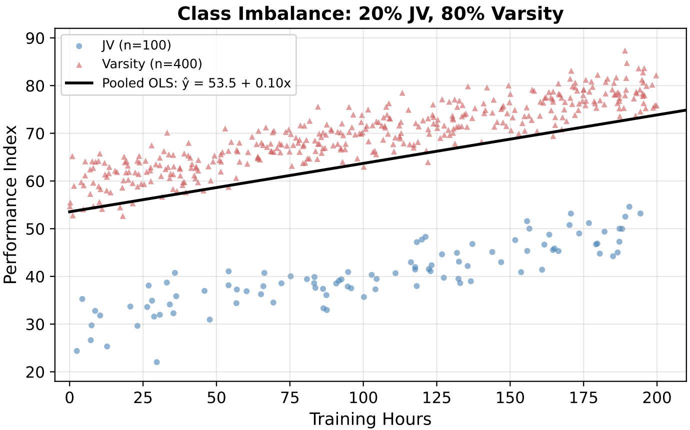

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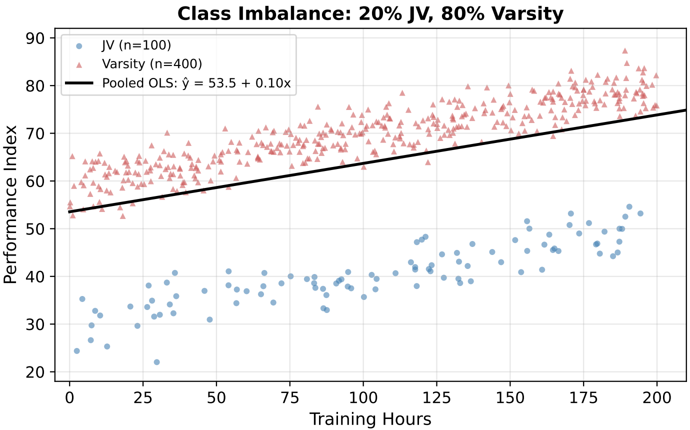

Slope ≈ correct. Intercept = **54** — pulled toward Varsity's true intercept (60). Now predictions are wrong for most JV players.

Class Imbalance: The Intercept Shifts

Sample	**OLS intercept**	**True JV** ($\alpha = 30$)	**True Var** ($\alpha = 60$)
80% JV	34	close	off by 26
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$\Longrightarrow$ Even when the slope is approximately correct, pooled OLS uses a **single intercept** that is wrong for every subgroup. The error depends on sample composition, which the researcher may not control.

A Separate Intercept per Unit

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This is the core idea behind **fixed effects**.

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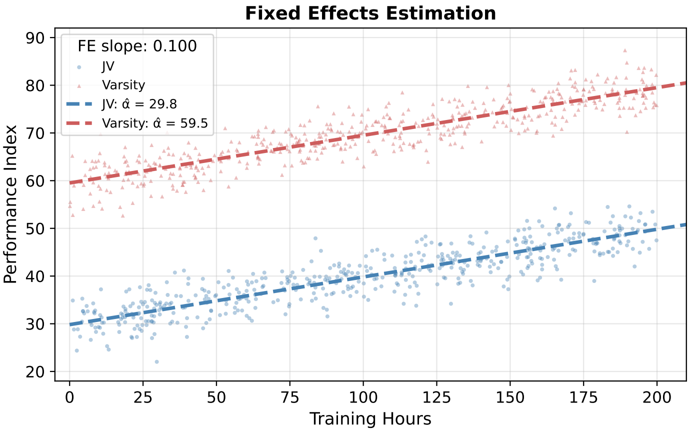
Key idea: FE estimates β using only *within-group* variation in x.
It asks: "Among JV players, do those who train more perform better?"

FE Estimation: The Result



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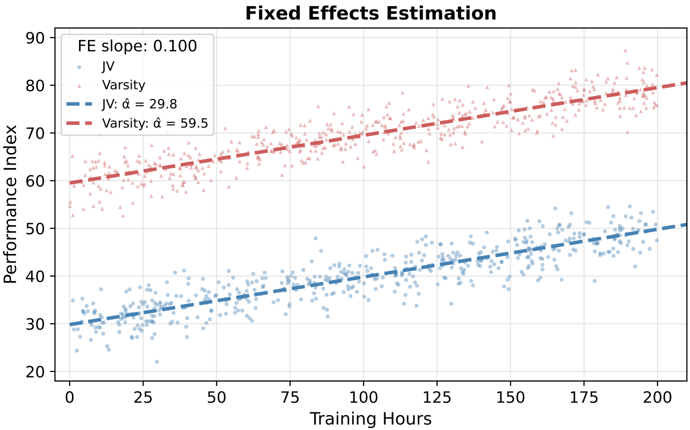
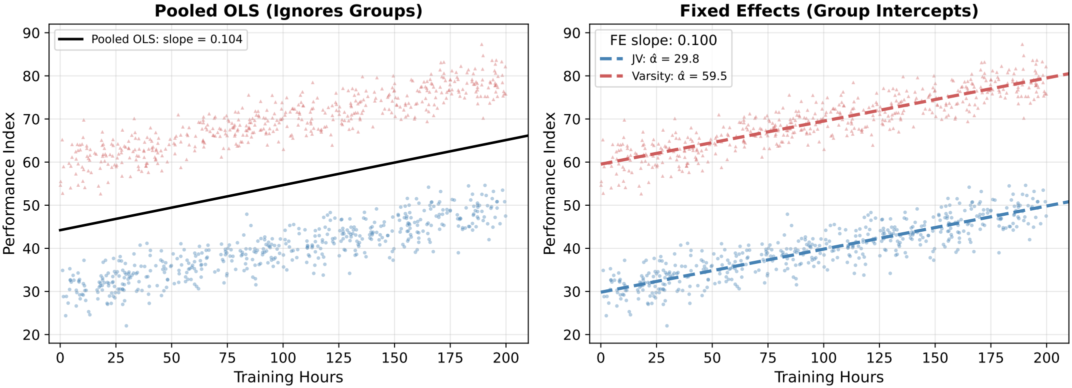


FE slope = **0.100** (true = 0.10). Intercepts: JV = **29.8** (true 30), Varsity = **59.5** (true 60).

Pooled OLS vs. Fixed Effects



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$\implies$ More groups = more unknowns = more data required.

Connection to Panel Data

Our training example maps directly to the panel data framework:

$$y_{it} = \alpha_i + \beta\, x_{it} + \varepsilon_{it}$$

- i = individual (trainee $\rightarrow$ firm, person, country)
- t = time period
- α_i = **individual fixed effect** (unobserved, time-invariant)

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1. **Least Squares Dummy Variable** (when we want all of the individual fixed effects):

$$y_{it} = \beta\, x_{it} + \sum_i \alpha_i D_i + \varepsilon_{it}$$

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Both give the **same** $\hat{\beta}$. Let's walk through the within estimator step by step.

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Notice that α_i survives averaging because it **doesn't vary over time**.

The Within Estimator: Step 2 $\rightarrow$ Subtract

Subtract the individual mean equation from the original:

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$\implies$ We have a simple regression with **no intercept** and no α_i.
Just run OLS on the demeaned data.

The Within Estimator: Step 3 $\rightarrow$ Estimate β

OLS on the demeaned regression gives:

$$\hat{\beta} = \frac{\text{Cov}(\ddot{x}_{it}, \ddot{y}_{it})}{\text{Var}(\ddot{x}_{it})} = \frac{\sum_i \sum_t (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i)}{\sum_i \sum_t (x_{it} - \bar{x}_i)^2}$$

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$\implies$ The covariance of the demeaned variables equals the covariance of the original variables:

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Why? Let's show it briefly.

Proof: Demeaning Doesn't Change Covariance

Let $c_i = \bar{x}_i$ and $d_i = \bar{y}_i$ (constants within group i). Then:

$$\begin{aligned}\text{Cov}(x_{it} - c_i,\ y_{it} - d_i) = \text{Cov}(x_{it},\ y_{it}) - \text{Cov}(x_{it},\ d_i) \\ - \text{Cov}(c_i,\ y_{it}) + \text{Cov}(c_i,\ d_i)\end{aligned}$$

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Within each group i, the means c_i and d_i are **constants**, so:

$$\text{Cov}(x_{it},\ d_i) = 0, \quad \text{Cov}(c_i,\ y_{it}) = 0, \quad \text{Cov}(c_i,\ d_i) = 0$$

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$\Longrightarrow$ The within estimator uses **only within-group variation**. All between-group differences are "absorbed" by the fixed effects.

Thank you!

jakeanderson@g.ucla.edu