Count Data Models

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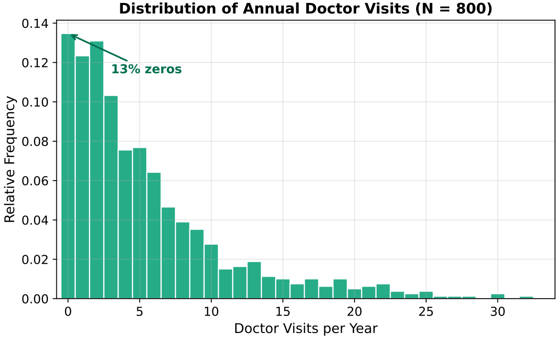
Outline

The Data

A health economist surveys **800 individuals** and records their **annual doctor visits**. Covariates include age, insurance status, and a health index (centered near 0; higher = healthier).



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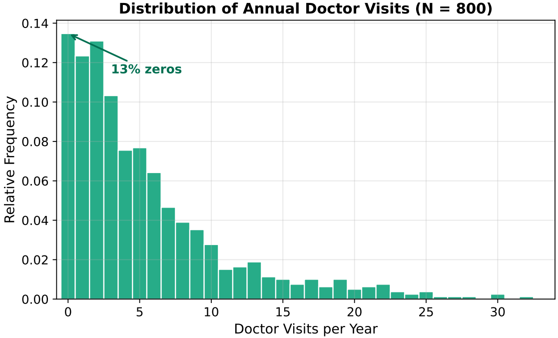

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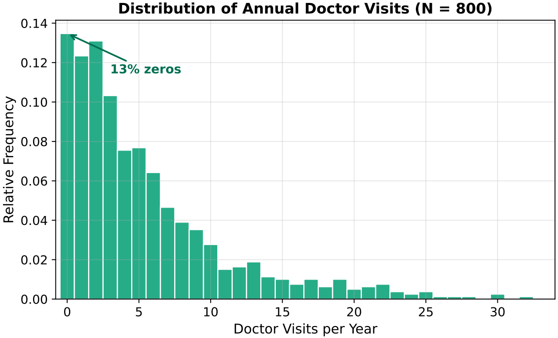

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The outcome is a **count**: non-negative integers (0, 1, 2, . . .). Right-skewed with a spike at zero. Mean = 5.7, but 13% have zero visits.

OLS Predictions on Count Data

Treat doctor visits as a continuous variable and regress on covariates:

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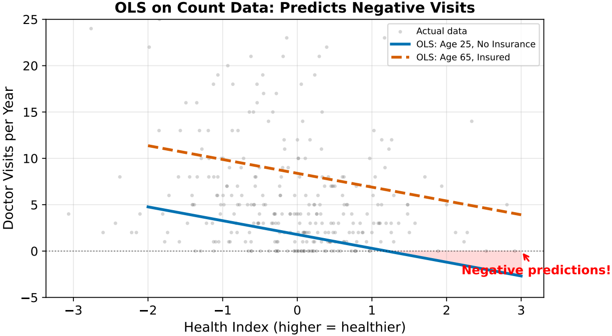

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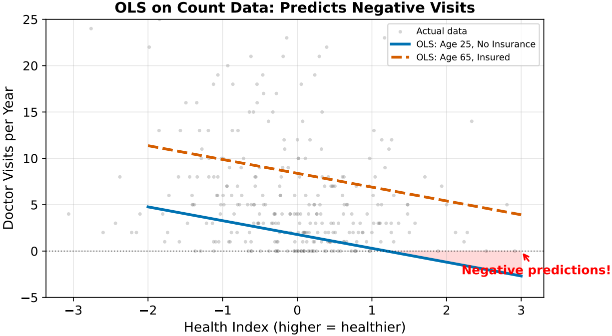

For a 25-year-old without insurance, OLS predicts **negative visits** once the health index exceeds about 1.5. Doctor visits cannot be negative.

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$\Longrightarrow$ We need a model built for count outcomes from the start.

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$\Longrightarrow$ Where can we find a probability distribution designed for non-negative integers?

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$\Longrightarrow$ Let's build a regression model on top of the Poisson distribution, just as logit builds on the logistic distribution.

Outline

The Poisson Distribution

A random variable Y follows a Poisson distribution with parameter $\mu > 0$ if:

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Example: if $\mu = 6$, then $P(Y = 0) = e^{-6} \approx 0.0025$ and $P(Y = 6) \approx 0.16$.

Poisson Regression: The Log Link

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Since $e^{(\cdot)} > 0$ for any input, **predicted counts are always positive**. This solves the negative-prediction problem.

Estimation: Maximum Likelihood

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$\implies$ The structure is identical to binary logit/probit MLE, just with a different distribution (Poisson instead of Bernoulli).

Interpreting Coefficients: Semi-Elasticities

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Example (Insurance, a dummy variable): if $\hat{\beta}_2 = 0.54$, then $e^{0.54} - 1 = 0.72$, so insured individuals have about 72% more visits.

Numeric Example: Predicted Visits

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Person A: 45 years old, insured, average health (Health $= 0$):

$$\ln(\hat{\mu}_A) = 0.50 + 0.017 \times 45 + 0.54 \times 1 + (-0.27) \times 0 = 1.805$$

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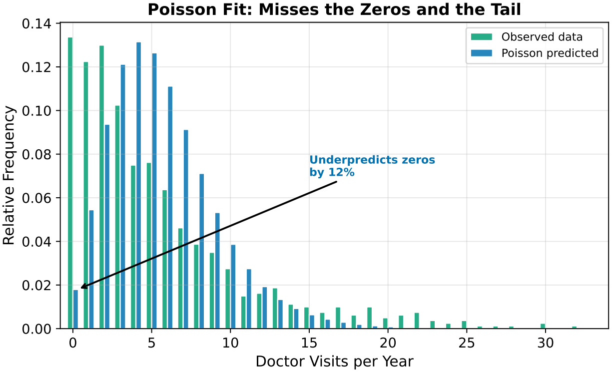
$\Longrightarrow$ Both predictions are positive. Compare to OLS, which predicted negative visits for Person B.

Poisson Fit to Our Data



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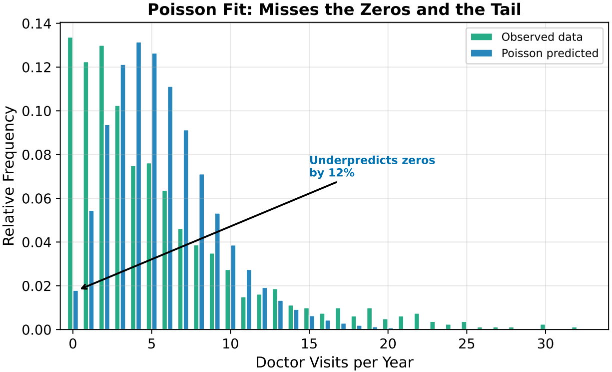

The Poisson predicts only 2% zeros; the data has 13%. It underpredicts zeros and underpredicts the right tail, concentrating too much mass in the middle. Why?

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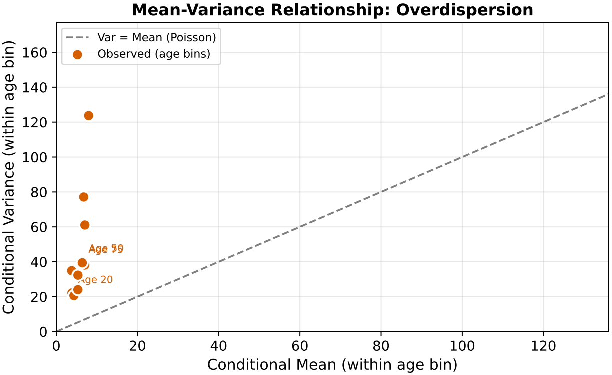
This is called **overdispersion**: more variability in the data than the Poisson distribution allows. It is extremely common with count outcomes.

Visualizing Overdispersion



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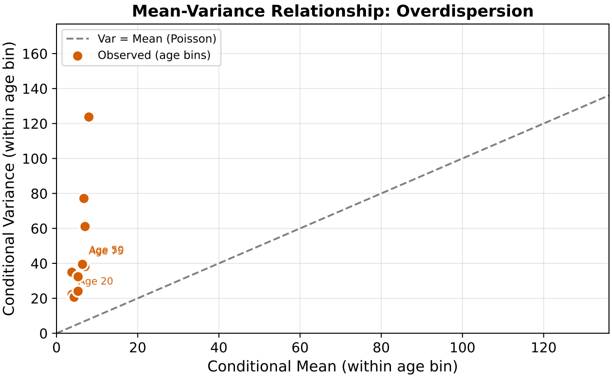

Every age bin lies **above** the 45-degree line. The variance grows faster than the mean, violating the Poisson assumption.

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$\implies$ With overdispersion, Poisson regression gives you the right answer with the wrong confidence.

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Can we keep the Poisson's log link but relax the variance constraint?

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$\Longrightarrow$ The **Negative Binomial** does exactly this: it generalizes the Poisson by adding one parameter.

Adding an Overdispersion Parameter

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$\implies$ Poisson is a special case of the Negative Binomial with $\alpha = 0$. The NB nests the Poisson.

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2. **NB probability formula** instead of Poisson. It uses a different formula to assign probabilities to each count value (software handles it)

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The Negative Binomial regression model specifies:

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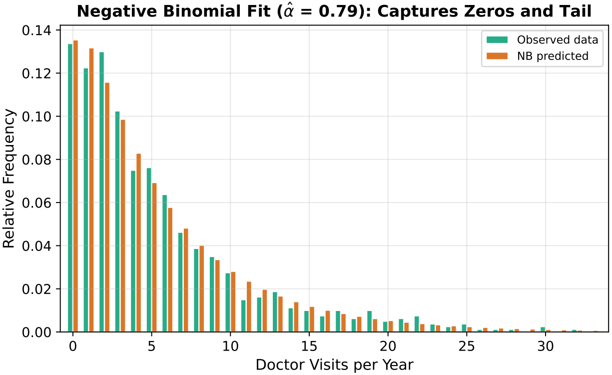
$\implies$ Coefficients have the **same semi-elasticity interpretation** as Poisson. The only change is allowing more variance.

NB Fit to Our Data



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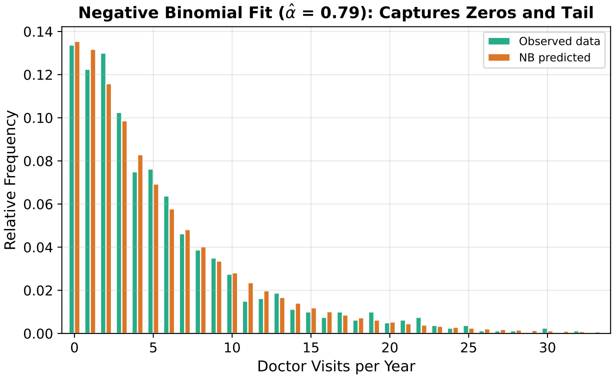
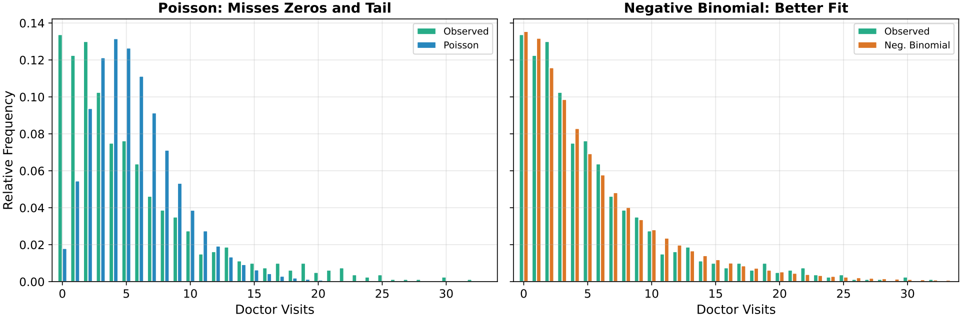


With an estimated $\hat{\alpha} = 0.79$, the Negative Binomial captures the spike at zero and the long right tail that Poisson missed.

Side-by-Side: Poisson vs. Negative Binomial



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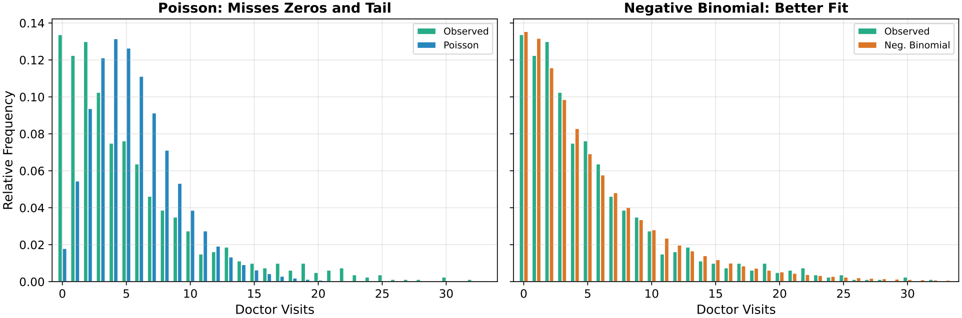

The Poisson (left) squeezes too much mass into the middle. The NB (right) spreads it out to match the data.

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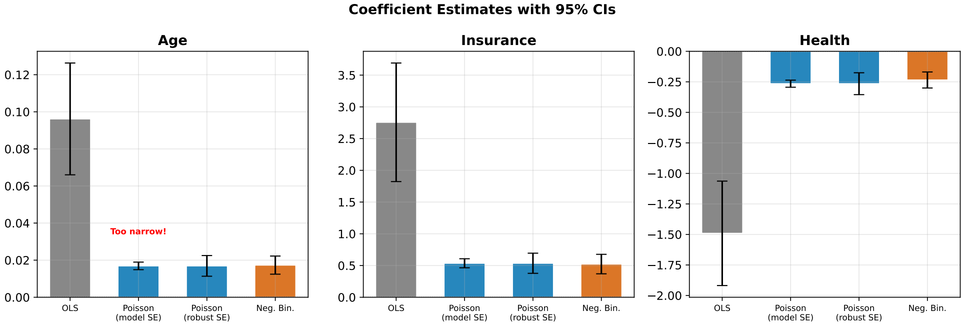
$\implies$ Strong evidence of overdispersion. The Poisson model is rejected in favor of NB.

Coefficient Estimates: OLS vs. Poisson vs. NB



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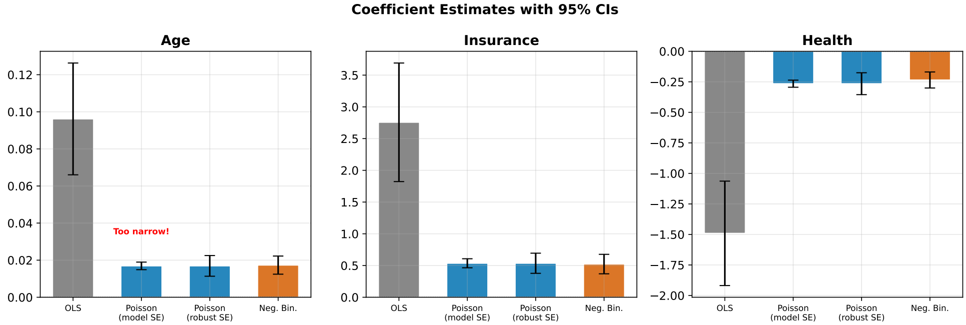

Poisson and NB give similar coefficient estimates, but Poisson model-based SEs are far too narrow. The NB SEs properly account for overdispersion.

Why Poisson SEs Are Too Small

	Poisson (model SE)	Poisson (robust SE)	NB
Age	0.001	0.003	0.003
Insurance	0.036	0.081	0.078
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$\Longrightarrow$ Both give CIs based on a consistent estimator of the true sampling variance, so coverage is correct asymptotically (typically wider than the under-dispersion-assuming Poisson CI).

Three-Model Comparison: OLS vs. Poisson vs. NB

	OLS	**Poisson**	**Neg. Binomial**
Predicted range	$(-\infty, +\infty)$	$(0, +\infty)$	$(0, +\infty)$
Variance assumption	constant	Var $= \mu$	Var $= \mu + \alpha\mu^2$
SE reliability (model-based)	heteroskedasticity biased	too small if overdispersed	correct if α well-estimated
Coefficient interpretation	level change (Δy per unit Δx)	semi-elasticity ($\approx \%\Delta y$)	semi-elasticity ($\approx \%\Delta y$)

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$\Longrightarrow$ Moving from OLS to Poisson solves the boundary problem; moving from Poisson to NB solves the variance problem.

Outline

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Approach	Variance structure	When to use
Poisson	$\text{Var} = \mu$	Mild or no overdispersion
Quasi-Poisson	$\text{Var} = \phi\,\mu$	Quick SE correction; no full likelihood
Neg. Binomial	$\text{Var} = \mu + \alpha\mu^2$	Full model; predictions, LR tests, AIC

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$\Longrightarrow$ In our data, NB captures the 13% zeros adequately. Zero-inflation would be needed if, say, 40% of the sample had zero visits.

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5. **If the outcome has a known upper bound** (e.g., number correct out of 10):

   $\Longrightarrow$ This is not a count model problem; consider binomial regression instead

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$\Longrightarrow$ Always start with Poisson, test for overdispersion, and upgrade to NB or robust SEs as needed.

Thank you!

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