Autocorrelation in Time-Series Regression

Detecting and Correcting Serially Correlated Errors

Jake Anderson

May 16, 2026

Outline

Time-Series Data and Persistent Shocks

In cross-sectional data, observations $i = 1, \ldots, n$ have no natural order. The standard OLS assumption $\text{Cov}(e_i, e_j) = 0$ for $i \neq j$ is plausible: two random households are unlikely to share an unobserved shock.

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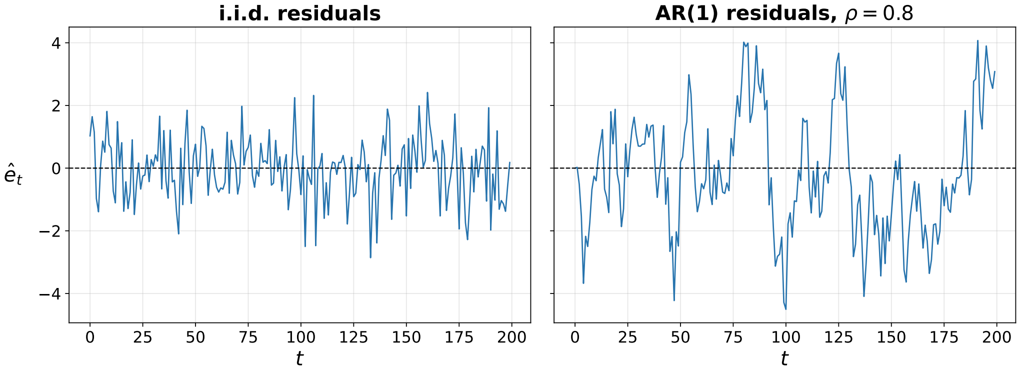
$\implies$ Errors are likely **serially correlated**: $\text{Cov}(e_t, e_{t-k}) \neq 0$ for some lag $k \geq 1$.

What Serial Correlation Looks Like

Compare two residual series from the same regression fit on simulated data:



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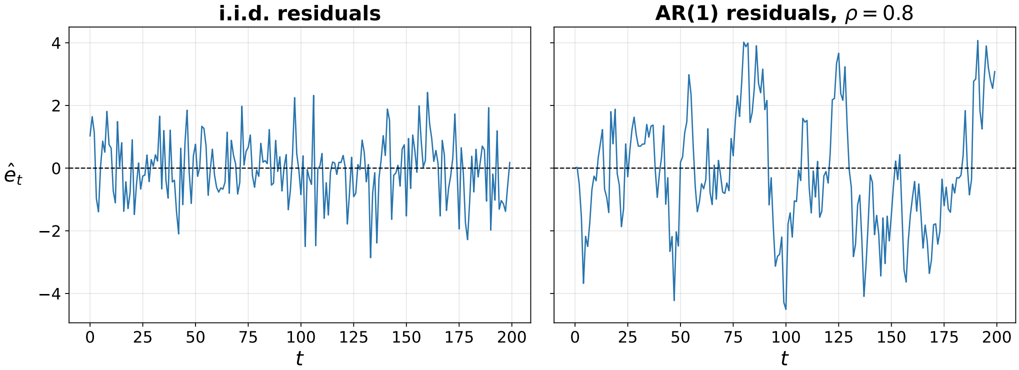

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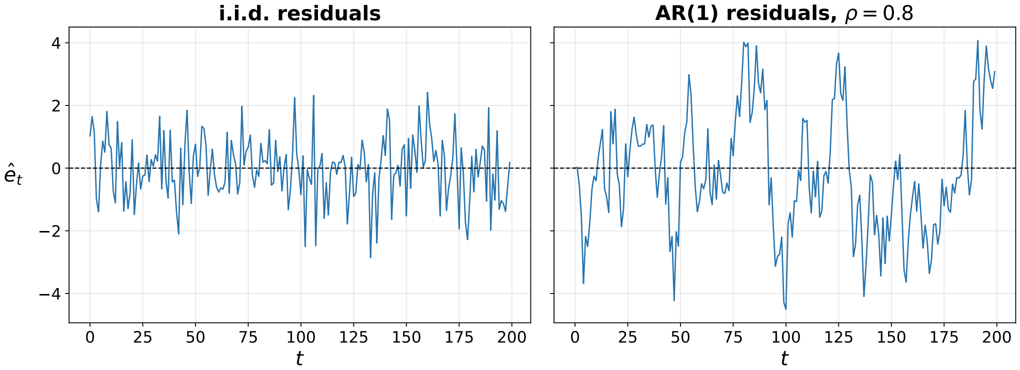

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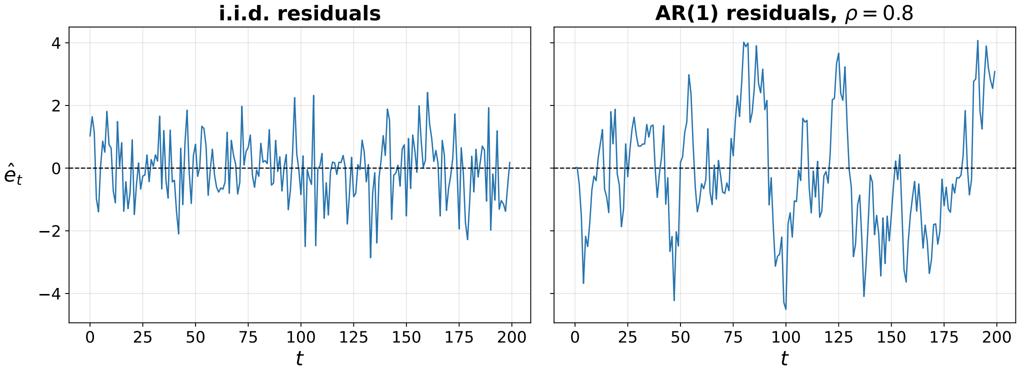

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Right: AR(1) errors with $\rho = 0.8$. Long positive and negative runs; the series *wanders*.

$\Longrightarrow$ Today's residual carries information about tomorrow's.

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Two Sources of Dynamics

With time-indexed data, today's outcome typically depends on:

- **Its own past:** $y_{t-1}, y_{t-2}, \ldots$ (momentum, habit, inertia).
- **The past of explanatory variables:** $x_{t-1}, x_{t-2}, \ldots$ (delayed effects, accumulation).

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$\Longrightarrow$ Specifying dynamics in y and x comes first; thinking about dynamics in the errors comes after.

Autoregressive (AR) Models

An AR(p) model regresses y_t on its own lags:

$$y_t = \alpha + \phi_1 \, y_{t-1} + \phi_2 \, y_{t-2} + \cdots + \phi_p \, y_{t-p} + u_t, \qquad u_t \sim \text{i.i.d.}(0, \sigma_u^2).$$

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Examples:

- GPA this quarter depends on GPA last quarter (study habits, cumulative mastery).
- Mood this hour depends on mood last hour (emotional inertia).
- Crime rate this year depends on crime rate last year (retaliation, network effects).

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Stationarity requires $|\phi_1 + \cdots + \phi_p| < 1$ (and stricter conditions for $p \geq 2$). Without it, shocks accumulate without bound.

Distributed Lag (DL) Models

A DL(q) model regresses y_t on contemporaneous and lagged values of x:

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$\Longrightarrow$ The sequence $\beta_0, \beta_1, \ldots, \beta_q$ traces out the **lag distribution** of x's effect on y. The sum $\sum_{s=0}^{q} \beta_s$ is the long-run multiplier.

ARDL Models: The Combination

An ARDL(p, q) model has both kinds of dynamics:

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$\implies$ The chapter's main interpretive payoff (lag weights, long-run multipliers, forecasting) lives inside the ARDL framework.

Errors That Remain Autocorrelated

Even after specifying an ARDL with as many lags of y and x as you think relevant, the errors e_t may *still* be autocorrelated. Reasons:

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Two parsimonious models for the error process:

$$\textbf{AR(1):} \quad e_t = \rho\, e_{t-1} + u_t, \qquad \textbf{MA(1):} \quad e_t = u_t + \theta\, u_{t-1},$$

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$\implies$ AR gives long decaying correlation; MA gives short abrupt correlation. The rest of this deck targets AR(1)-type structure (the more common case) with tests that also pick up MA(1) and higher orders.

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OLS Under Autocorrelation: The Good News

Suppose the regressors are strictly exogenous: $\mathrm{E}[e_t \mid x_1, \ldots, x_T] = 0$ for all t. Then even with serially correlated errors, OLS is still:

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$\implies$ The coefficient estimates themselves are fine (in the strictly exogenous case). The problem is the standard error.

Where the SE Formula Goes Wrong

For a single-regressor model, the true sampling variance of the slope is

$$\mathrm{Var}(\hat{\beta}_1) = \frac{1}{\left[\sum_t (x_t - \bar{x})^2\right]^2} \, \mathrm{Var}\left(\sum_t (x_t - \bar{x}) \, e_t\right).$$

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$\Longrightarrow$ OLS standard errors are **inconsistent** under autocorrelation. *Software handles the multi-regressor case the same way (matrix sandwich form).*

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Negative autocorrelation: the reverse can happen, but is rare in applied work.

Visualizing the Damage

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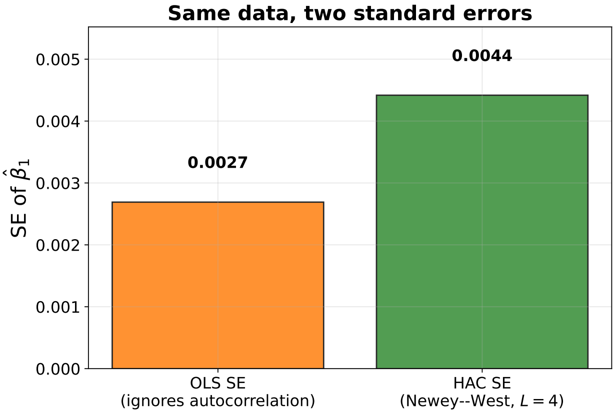

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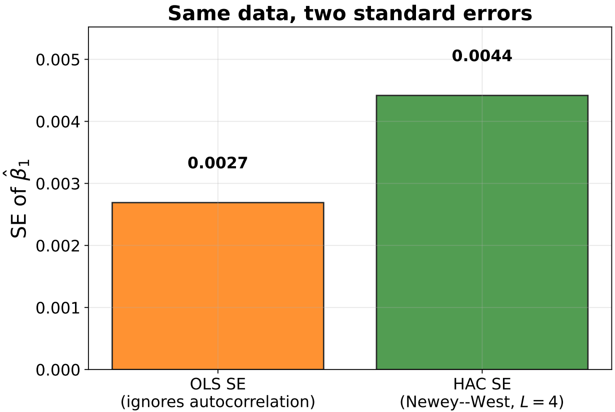

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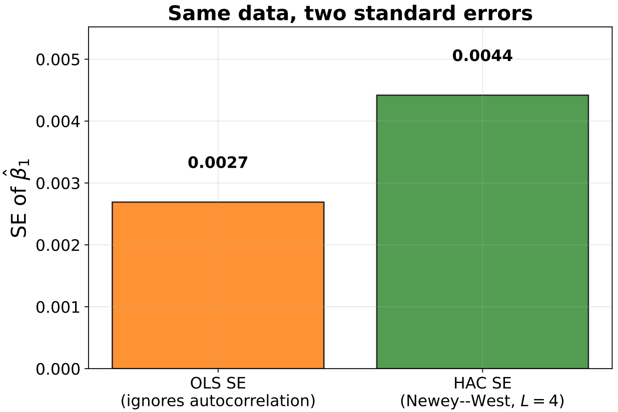

The HAC bar (right) is consistent; the OLS bar (left) is what you would report if you ignored the autocorrelation.
$\Longrightarrow$ Using OLS SEs would shrink the CI by roughly half and inflate the t-statistic accordingly.

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Implication for the DW test (coming up): DW assumes strict exogeneity and is *not* valid when a lagged dependent variable is a regressor.

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Visual Detection: Three Pictures

Before running a formal test, look at the residuals. Three plots cover most cases:

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None of these gives a p-value, but together they tell you what *kind* of autocorrelation you have, which guides which formal test to run.

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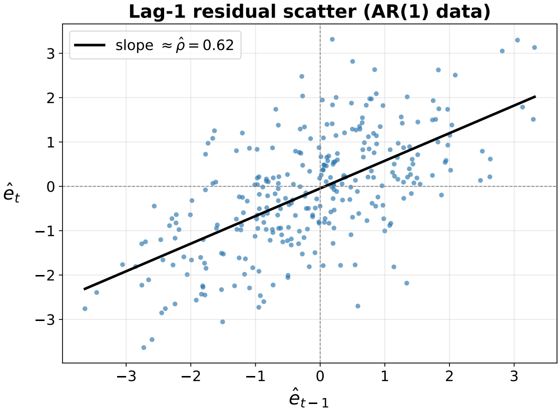

The Lag-1 Scatter

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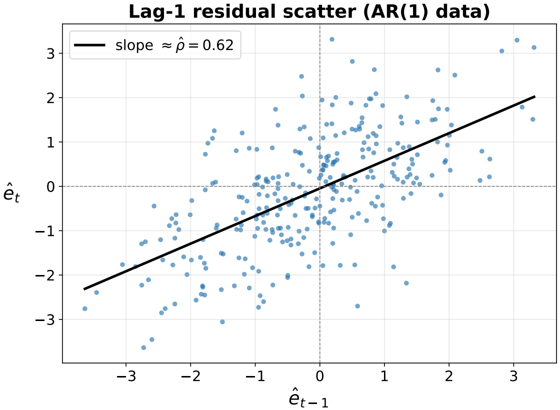

A line through this cloud has slope approximately $\hat{\rho}$. Upward tilt $\implies$ positive autocorrelation at lag 1; the visual analogue of regressing $\hat{e}_t$ on $\hat{e}_{t-1}$.

The Sample Autocorrelation Function

Define the sample autocorrelation at lag k:

$$r_k = \frac{\sum_{t=k+1}^{T} (\hat{e}_t - \bar{\hat{e}})(\hat{e}_{t-k} - \bar{\hat{e}})}{\sum_{t=1}^{T} (\hat{e}_t - \bar{\hat{e}})^2}.$$

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Under the null of i.i.d. residuals and for large T,

$$r_k \approx \mathcal{N}\left(0,\ \tfrac{1}{T}\right) \quad \Longrightarrow \quad \text{individual } r_k \text{ is ``significant'' if } |r_k| > \frac{1.96}{\sqrt{T}}.$$

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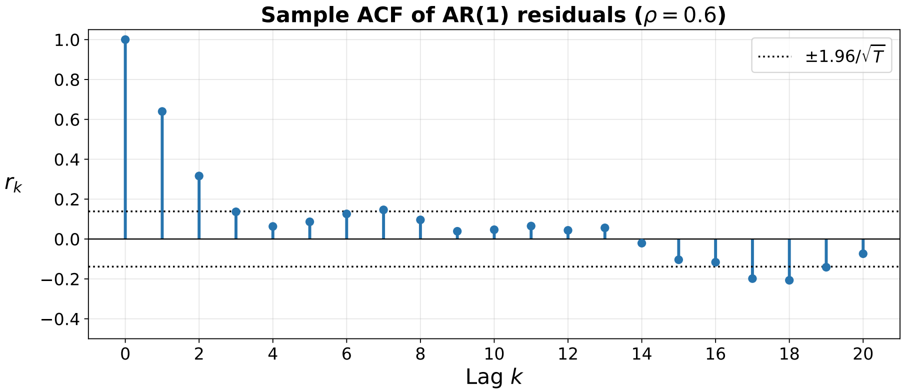
Read it like this:

- Bar at lag 1 outside the band $\Longrightarrow$ probable AR(1) or MA(1).
- Bars at lags 1 and 2 outside the band $\Longrightarrow$ probable AR(2) or longer.
- Bar at lag 12 (monthly) or lag 4 (quarterly) outside the band $\Longrightarrow$ seasonal autocorrelation.

Example ACF: AR(1) Residuals



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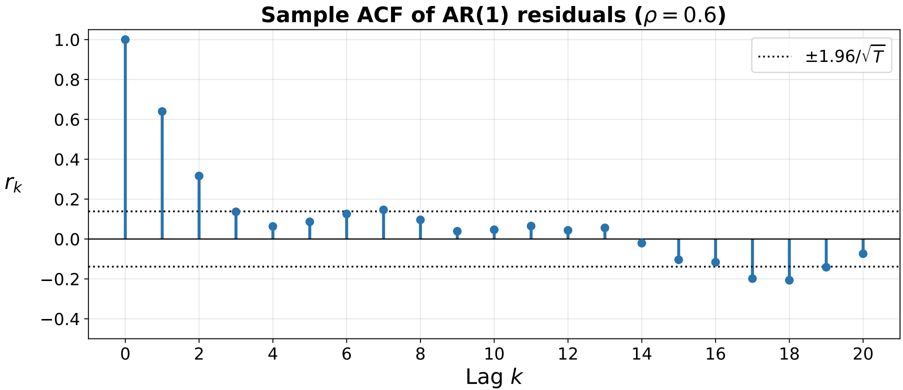

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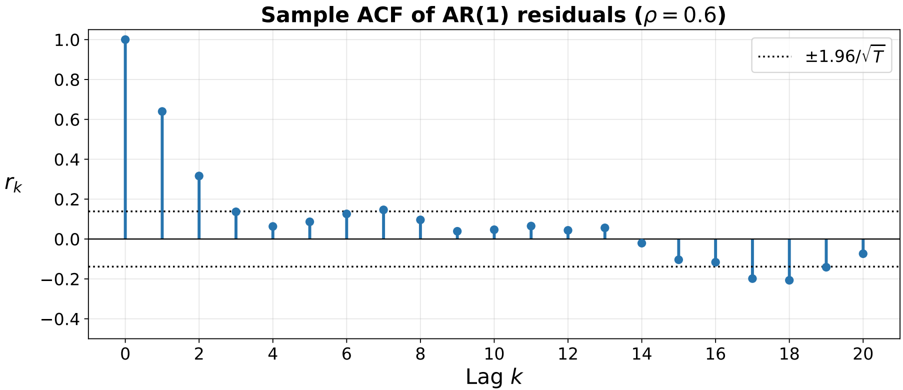

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$\Longrightarrow$ A formal test with $p = 1$ (BG) or with the AR(1) alternative (DW) is the natural next step.

Formal Tests for Autocorrelation

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$\Longrightarrow$ We work through each test in turn.

Outline

DW Test: The Idea

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The Durbin–Watson statistic measures how different $\hat{e}_t$ is from $\hat{e}_{t-1}$ on average. Define

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A bit of algebra (expand the square and use $\sum \hat{e}_t^2 \approx \sum \hat{e}_{t-1}^2$ for large T) gives the useful approximation

$$DW \approx 2(1 - \hat{\rho}), \qquad \text{where } \hat{\rho} = \frac{\sum_{t=2}^{T} \hat{e}_t \hat{e}_{t-1}}{\sum_{t=1}^{T} \hat{e}_t^2}.$$

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- $\hat{\rho} \approx 0 \implies DW \approx 2$ (no autocorrelation).
- $\hat{\rho} \approx 1 \implies DW \approx 0$ (strong positive autocorrelation).
- $\hat{\rho} \approx -1 \implies DW \approx 4$ (strong negative autocorrelation).

Hypotheses and the Inconclusive Zone

Null and alternative (one-sided, against positive autocorrelation):

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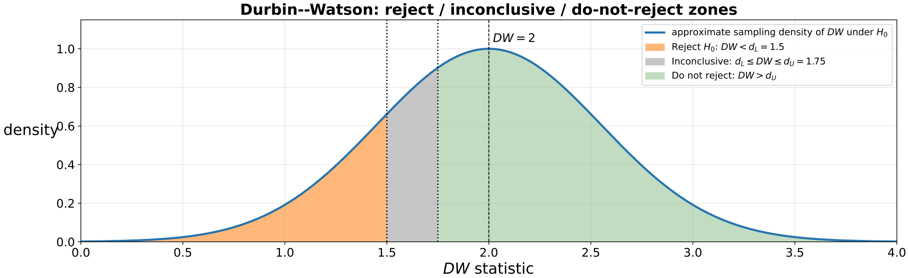
Unlike a t or χ^2 test, the exact distribution of DW depends on the particular values of the regressors in your sample. Durbin and Watson tabulated **two** critical values, d_L (lower) and d_U (upper):



The Decision Rule

For the lower-tail test (against $\rho > 0$):

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Why two critical values? The sampling distribution of DW under H_0 shifts left or right depending on the regressors. d_L and d_U bracket the worst-case and best-case designs $\Longrightarrow$ any DW between them is consistent with both rejection and non-rejection for *some* design.

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$\Longrightarrow$ The inconclusive zone is the price of avoiding a critical value that depends on the specific regressors used.

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$\Longrightarrow$ DW is fine for a quick screen. For anything beyond AR(1), use Breusch–Godfrey (next section).

Outline

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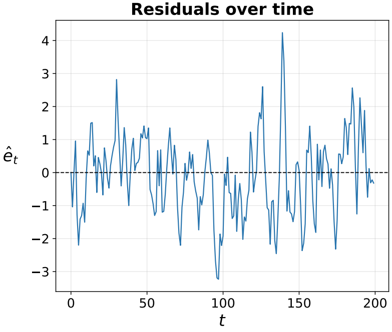
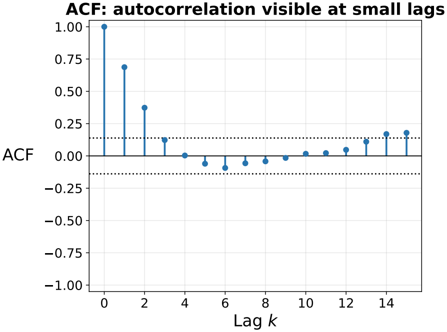
Premise: If autocorrelation is present, OLS residuals at time t are related to residuals at earlier times. The BG test regresses $\hat{e}_t$ on its own lags (and the original regressors) and asks whether the lag coefficients are jointly zero.





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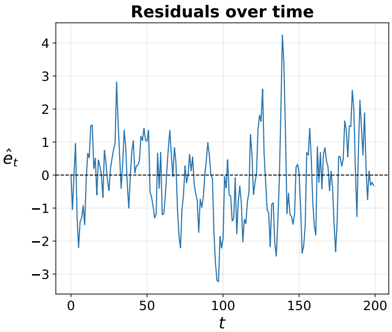
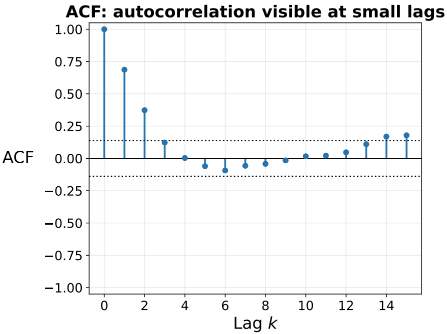
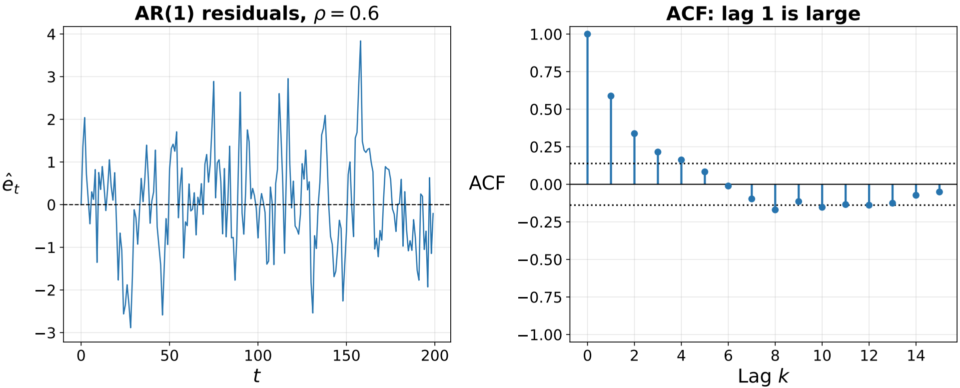




If past residuals predict current residuals $\implies$ reject no-autocorrelation.

When the BG Test Works: AR(1) Errors



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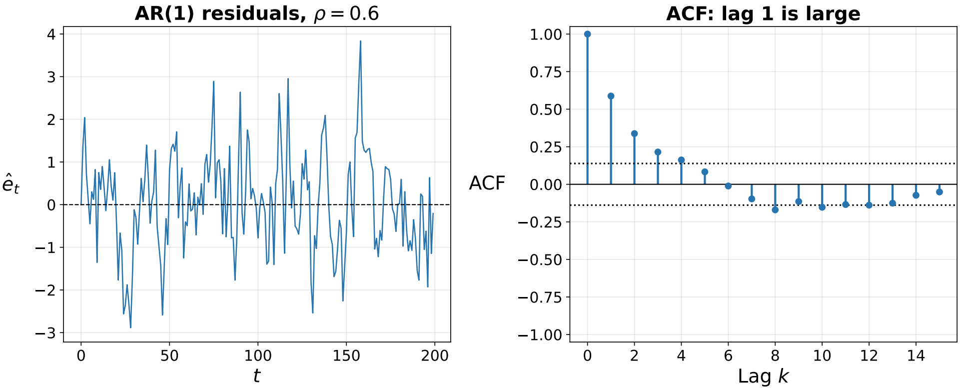

Residuals follow $e_t = 0.6\, e_{t-1} + u_t$. The ACF spike at lag 1 is exactly what BG with $p = 1$ is built to catch.

BG Test successfully detects autocorrelation!

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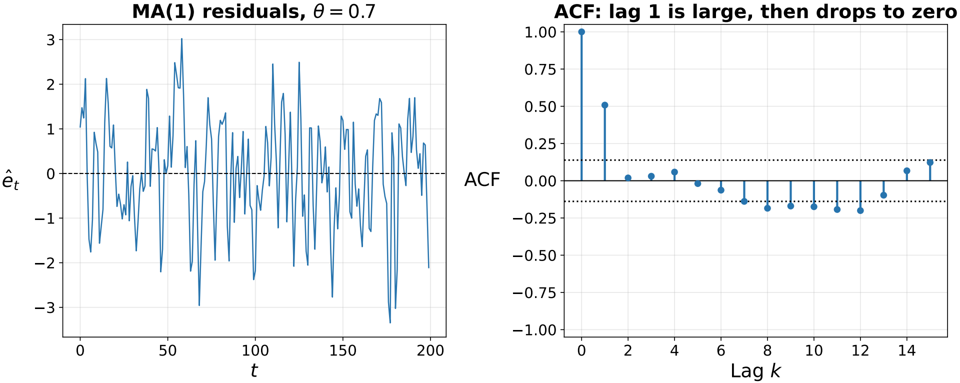

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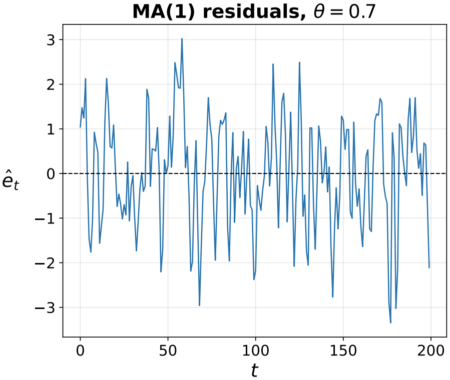
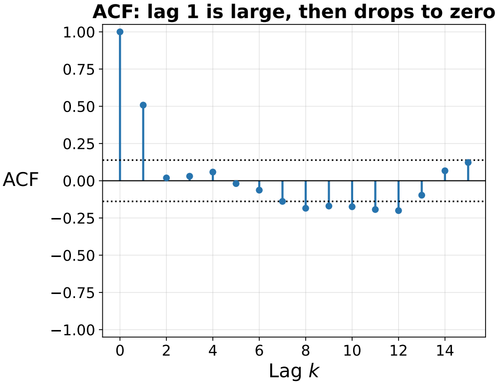
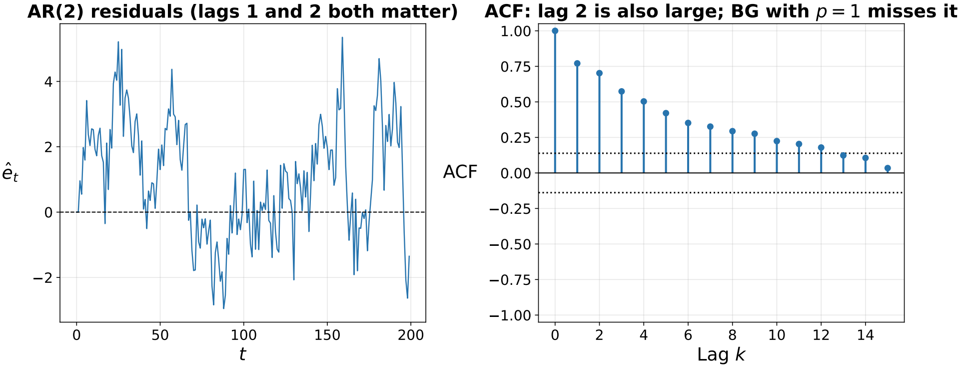




Residuals follow $e_t = u_t + 0.7\, u_{t-1}$. ACF lag 1 is large; lags 2+ drop to zero. BG with $p = 1$ catches this.

BG Test successfully detects autocorrelation!

When the BG Test Fails: Wrong Lag Order



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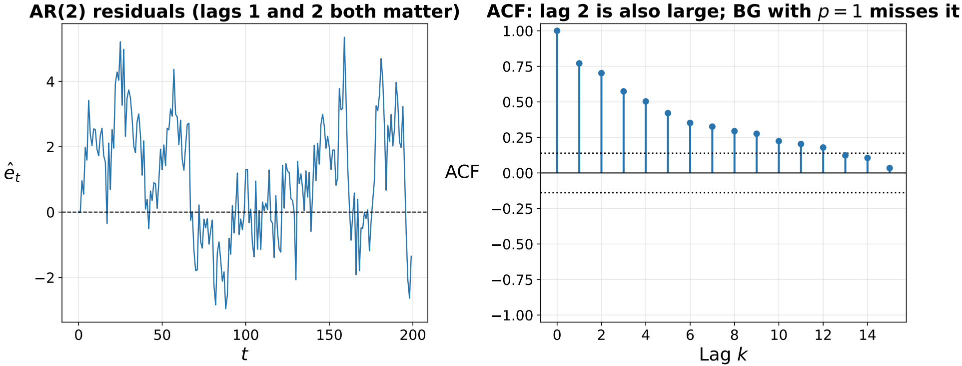
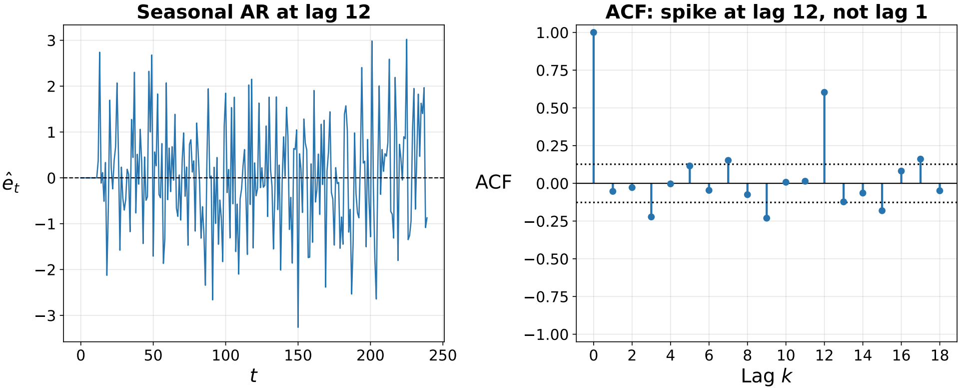


Residuals follow $e_t = 0.5\, e_{t-1} + 0.4\, e_{t-2} + u_t$ (AR(2)). Lag 2 carries real signal but BG with $p = 1$ ignores it. Either pick a larger p, or the test understates the autocorrelation.

BG Test fails to detect this kind of autocorrelation

When the BG Test Fails: Seasonal Autocorrelation



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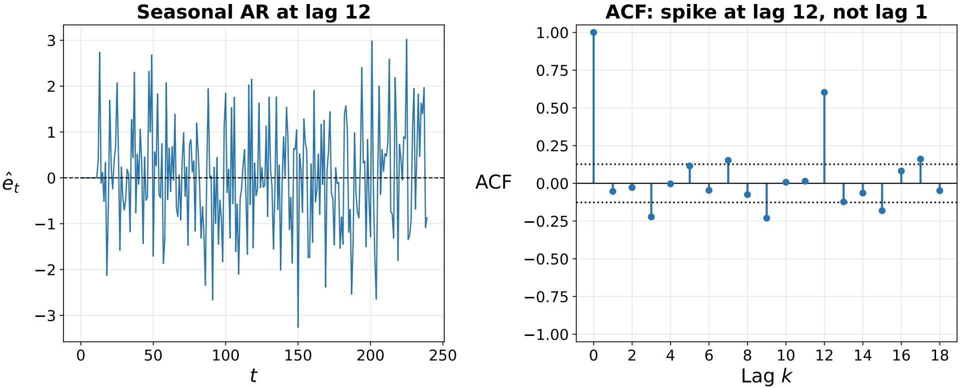
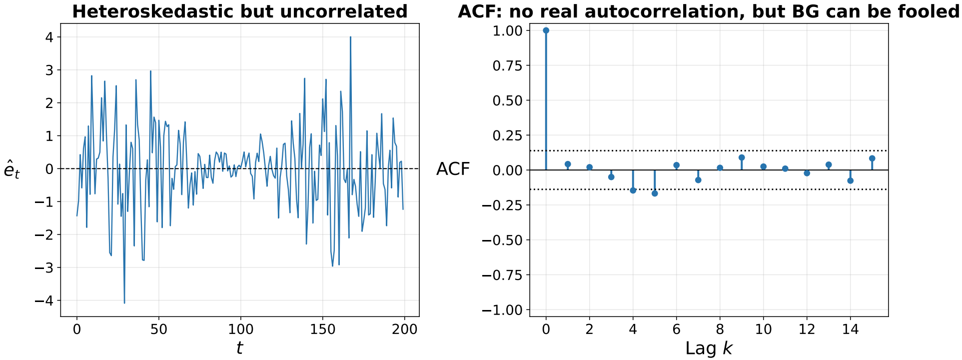


Residuals follow $e_t = 0.7\, e_{t-12} + u_t$ (seasonal AR at lag 12). The ACF spike is at lag 12; lags 1–11 are roughly zero. BG with $p \leq 11$ misses it entirely.

BG Test fails to detect this kind of autocorrelation

When the BG Test Fails: Heteroskedastic but Uncorrelated



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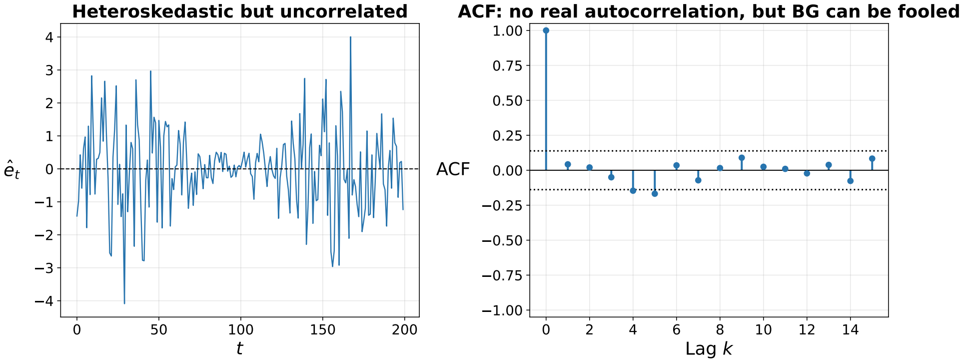

Residuals are independent but their variance changes over time. The ACF should be flat, but the slow variance drift can occasionally produce spurious lag-k correlations and lead BG to falsely reject.

BG Test can be misled by non-autocorrelation problems

BG Test: Procedure

Auxiliary regression: regress the OLS residual on the original regressors *and* on p lagged residuals.

$$\hat{e}_t = \alpha_0 + \alpha_1 x_t + \rho_1 \hat{e}_{t-1} + \rho_2 \hat{e}_{t-2} + \cdots + \rho_p \hat{e}_{t-p} + v_t$$

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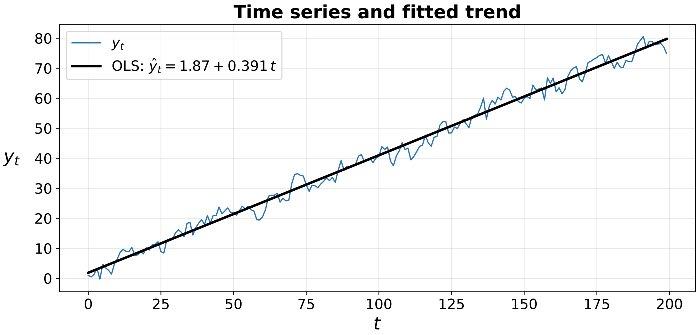
The degrees of freedom equal p, the number of *lagged-residual* terms in the auxiliary regression. **The original regressors and intercept are not counted.**

Choosing p: rule of thumb is $p = 1$ for clean monthly/yearly data, $p = 4$ for quarterly, $p = 12$ for monthly seasonal. When unsure, run several p's.

Example: Trending Series with AR(1) Errors

We have a time series y_t (e.g., a monthly sales index) over $n = 200$ periods. We model y_t as a linear trend in t:

$$y_t = \beta_0 + \beta_1 t + e_t$$



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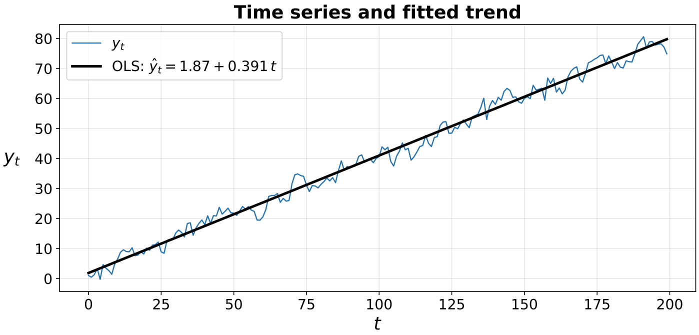

The fitted line captures the trend, but consecutive residuals around it tend to have the same sign: large positive deviations cluster together, large negative deviations cluster together. That is the visual signature of positive autocorrelation. We run BG with $p = 1$ to test for first-order autocorrelation.

Step 1: Run OLS

Coefficients

	Estimate	Std. Error	t-value	$\Pr(>\|t\|)$
(Intercept)	1.8656	0.3311	5.63	$< 2 \cdot 10^{-16}$ ***
t	0.3914	0.0029	135.98	$< 2 \cdot 10^{-16}$ ***

Residual standard error: 2.35 on 198 degrees of freedom
Multiple R^2*:* 0.9894 *Adjusted* R^2*:* 0.9894
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The trend is extremely significant under OLS *assumptions*. **But are the residuals independent over time?** BG will tell us.

Step 2: State the Hypothesis

The BG test asks whether $\hat{e}_t$ depends on its own past values, after controlling for t.

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Test statistic:

$$\text{BG} = (n - p) \cdot R^2_{\text{aux}} \sim \chi^2(p) \text{ under } H_0$$

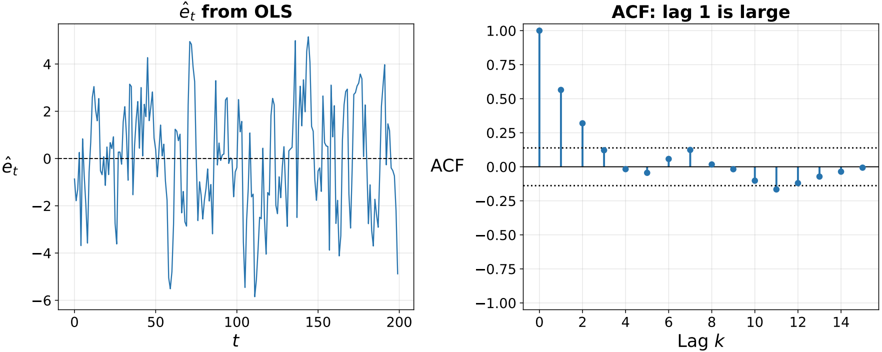
where p is the number of *lagged-residual* terms (the regressors and intercept are not counted). Here $p = 1$.

Step 3: Auxiliary Regression

Compute residuals $\hat{e}_t$ and regress on t plus one lag of itself:

$$\hat{e}_t = \alpha_0 + \alpha_1\, t + \rho_1\, \hat{e}_{t-1} + v_t$$

Estimated: $\hat{\rho}_1 = 0.577, \quad R^2_{\mathrm{aux}} = 0.326.$



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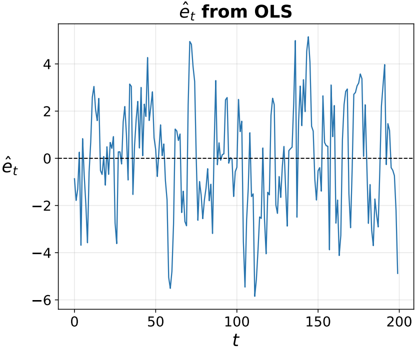

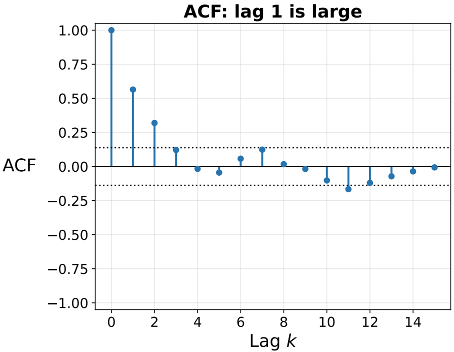

The residual time series wanders rather than scattering randomly; the ACF lag-1 spike is exactly what BG is built to catch.

Step 4: Compute the Test Statistic

Plug in:

$$\begin{aligned}\text{BG} &= (n-p)\cdot R^2_{\text{aux}} \\ &= (200-1)\cdot 0.326 \\ &= 199\cdot 0.326 \\ &= \boxed{64.92}\end{aligned}$$

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Compare to the critical value of $\chi^2(1)$ at $\alpha = 0.05$.

Step 5: Compare to χ^2 Critical Value

df	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$
1	2.706	**3.841**	5.024	6.635
2	4.605	5.991	7.378	9.210
3	6.251	7.815	9.348	11.345
4	7.779	9.488	11.143	13.277
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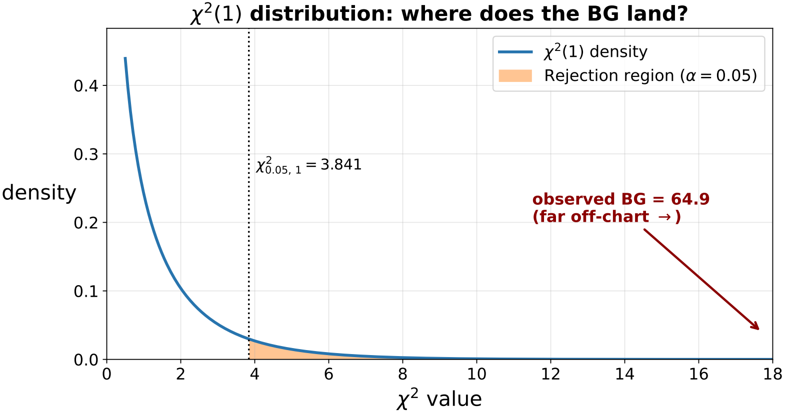
Reject H_0 if BG > 3.841.

Step 6: Visualize the Decision



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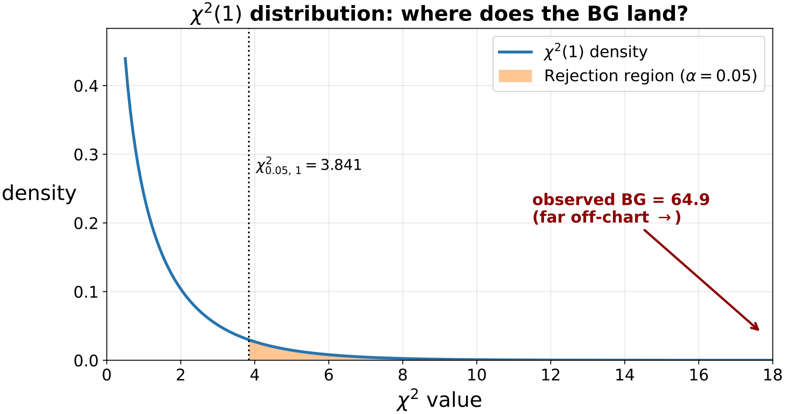

The orange region is the upper 5% tail of $\chi^2(1)$. BG $= 64.92$ is far past the right edge; p-value is essentially zero.

Step 7: Conclusion

Decision: $\text{BG} = 64.92 \gg 3.841 \implies$ **reject** H_0 at $\alpha = 0.05$ (and at any conventional level).

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Implications:

- OLS coefficients ($\hat{\beta}_0 = 1.87$, $\hat{\beta}_1 = 0.391$) remain unbiased and consistent.
- OLS standard errors (e.g., 0.0029 on the trend) are **inconsistent**; the gap does not shrink with n.
- Use *HAC* (Newey-West) standard errors, or model the AR structure directly via GLS.

Outline

Test-and-Fix vs Correct-the-SE

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$\Longrightarrow$ Newey–West HAC SEs are the workhorse: they require no parametric model for the error dynamics and are the applied-econometrics default.

The HAC Idea

Recall the diagonal-plus-off-diagonal decomposition (from the consequences section):

$$\mathrm{Var}\left(\sum_t (x_t - \bar{x})\, e_t\right) = \underbrace{\sum_t (x_t - \bar{x})^2\ \mathrm{Var}(e_t)}_{\text{White-style term}} + \underbrace{2\sum_{t<s} (x_t - \bar{x})(x_s - \bar{x})\ \mathrm{Cov}(e_t, e_s)}_{\text{autocorrelation term}}.$$

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Why not sum all $T - 1$ **lags?** Sample autocovariances at far lags are noisy. Newey–West truncates the sum at a chosen **bandwidth** L and applies triangular down-weighting to the lags that are kept (the Bartlett kernel).

The Newey–West Estimator: Schematic

Newey–West estimates the long-run variance as two pieces:

$$\widehat{S}_L = \underbrace{(\text{White-style sum})}_{\text{diagonal: } \hat{e}_t^2 \text{ terms}} + \underbrace{(\text{weighted sum of lag-}k\text{ autocovariances, } k = 1, \ldots, L)}_{\text{autocorrelation correction}}.$$

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$\Longrightarrow$ The estimator combines a heteroskedasticity correction (the White part) with an autocorrelation correction. Software handles the explicit weights and the multi-regressor matrix form.

Choosing the Bandwidth L

The bandwidth trades bias against variance:

- L too small $\implies$ omits real autocovariances; SEs still inconsistent.
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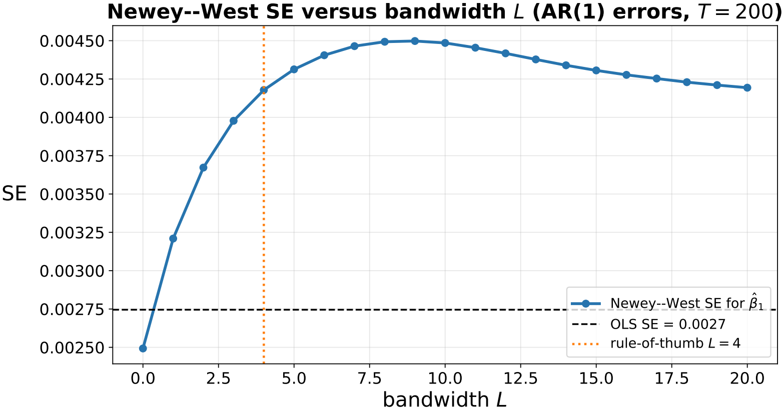
In practice:

- Software picks a sensible default (typically $L \approx 4$ for sample sizes near $T = 100$, scaling slowly with T).
- Report SEs for a small range of L values as a sanity check.
- If the SE is stable across $L \in \{2, 4, 8\}$, you're fine. If it drifts upward as L grows, the autocorrelation extends further than the default captures.

Bandwidth Sensitivity in Practice



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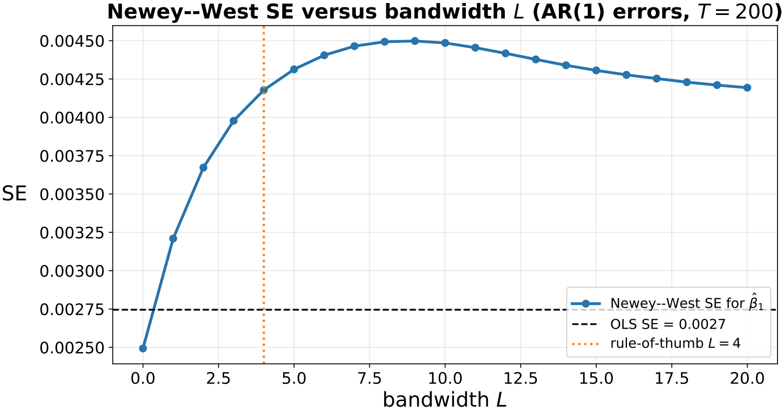

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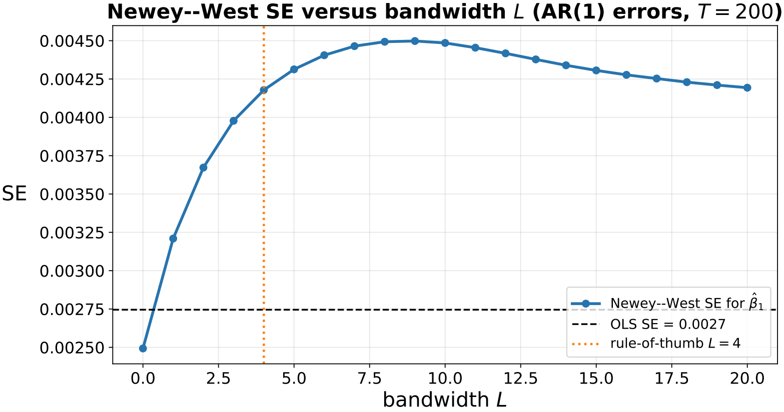

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$\Longrightarrow$ Read the value off the plateau, not off $L = 0$ and not off the largest L available.

Newey–West in R

Using the sandwich and lmtest packages:

```
library(sandwich) library(lmtest)
model <- lm(y   x, data = ts_data)
# Auto bandwidth coeftest(model, vcov. = NeweyWest(model))
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- Point estimate $\hat{\beta}$ is unchanged.
- Only the SE changes (and therefore CI, t, p).
- Reporting OLS and HAC SEs side-by-side is standard practice.

Outline

Summary: Detect, Test, Correct

Tool	When to Use	Limitation
Time plot of $\hat{e}_t$	Always: first look at the residuals	Subjective; no p-value
Lag-1 scatter, ACF	Diagnosing the kind of autocorrelation	Suggestive, not formal
Durbin–Watson	Quick test for AR(1) errors with strictly exogenous regressors	Only AR(1); inconclusive zone; fails with lagged y
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The decision flow:

1. Run OLS. Plot residuals and ACF.
2. Run BG (preferred) or DW. If significant $\implies$ OLS SEs are unreliable.
3. Report Newey–West HAC SEs alongside (or instead of) OLS SEs.

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- **Spurious regression and cointegration:** two trending series can produce significant OLS slopes even when they share no economic relationship. The tools developed here (DW, BG, HAC) help diagnose the problem, but the fix is differencing or cointegration analysis.

The minimum you should leave with:

- Autocorrelation breaks OLS SEs, not (usually) OLS point estimates.
- Always test in time-series regressions; BG dominates DW outside of textbook AR(1) cases.
- HAC standard errors are the default correction $\iff$ you can keep OLS coefficients and still report valid t-statistics.

Thank you!

jakeanderson@g.ucla.edu

Time Series: Dynamic Models and Autocorrelation

Modeling Temporal Dependence in Economic Data

Jake Anderson

May 16, 2026

Outline

Cross-Section vs Time Series

Cross-sectional data: observations are independent draws from a population.

- Randomly sample 500 workers $\implies$ one worker's wage tells you nothing about another's

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$\implies$ The i.i.d. assumption fails. We need new tools that account for **temporal dependence**.

Consequences of Temporal Dependence

When observations are correlated over time:

1. Standard OLS **standard errors are wrong** (usually too small)
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This lecture covers:

- How to characterize temporal dependence (stationarity, ACF)
- Models that exploit it (AR, ARDL)
- How to detect it in regression residuals (Breusch-Godfrey)
- How to forecast with it

Covariance Stationarity: Definition

A time series $\{Y_t\}$ is **covariance stationary** if:

1. **Constant mean:** $E(Y_t) = \mu$ for all t
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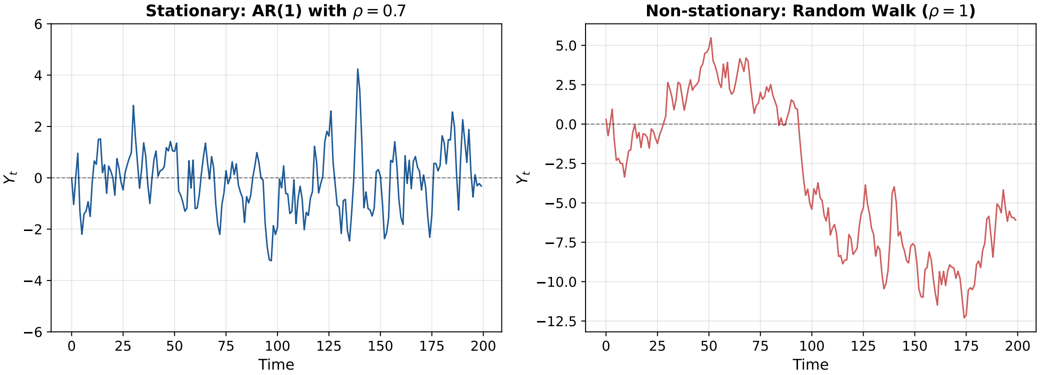
$\implies$ Stationarity ensures that estimated statistical properties are meaningful and stable.

Stationary vs Non-stationary: Visual Comparison



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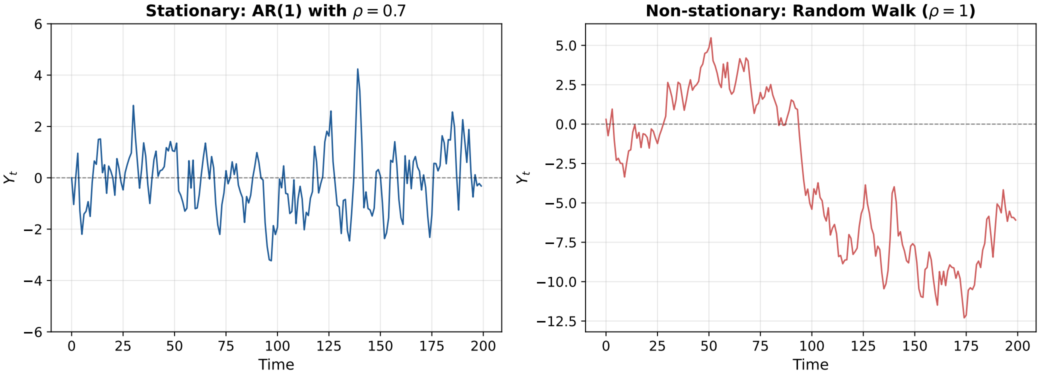

Left: mean-reverting; always pulled back toward zero. **Right:** wanders without bound; no tendency to return.

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$\implies$ Shocks accumulate forever. The series has no tendency to revert to any level.

Example: Stock prices are approximately random walks. That is why financial economists work with *returns* (which are approximately stationary) rather than price levels.

Measuring Temporal Dependence: The ACF

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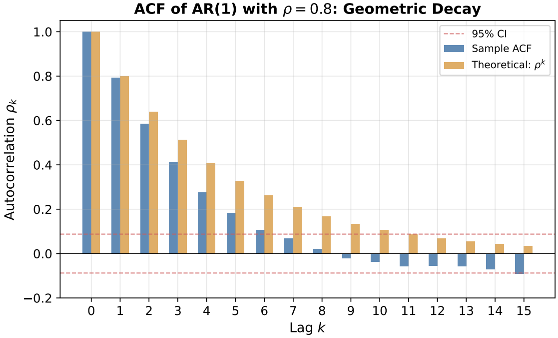
The **sample autocorrelation**:

$$r_k = \frac{\sum_{t=k+1}^{T}(Y_t - \bar{Y})(Y_{t-k} - \bar{Y})}{\sum_{t=1}^{T}(Y_t - \bar{Y})^2}$$

Reading an ACF Plot



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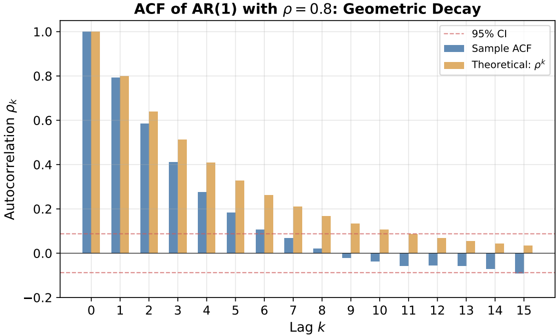

How to read it:

- Bars beyond the dashed lines = statistically significant autocorrelation
- For an AR(1), the ACF decays **geometrically**: $\rho_k = \rho^k$
- If residuals from a model show significant ACF spikes $\implies$ the model is missing dynamics

Autoregressive Models: AR(p)

An **AR(p) model** says today's value depends on its own past p values:

$$Y_t = \delta + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \cdots + \theta_p Y_{t-p} + v_t$$

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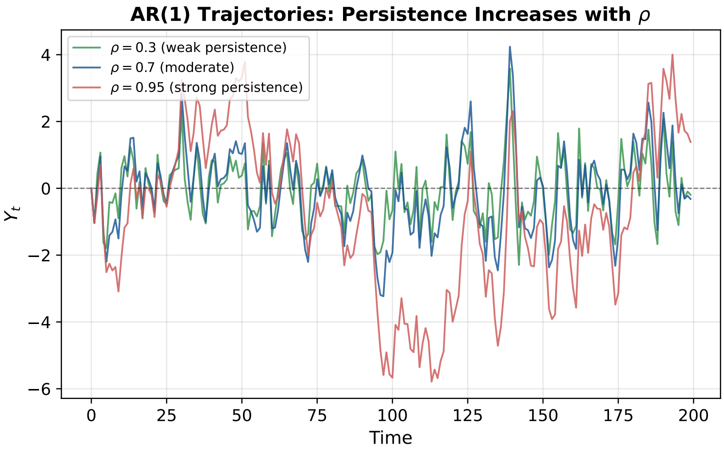
AR(2): $Y_t = \delta + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + v_t$

- Allows richer dynamics: oscillations, humps

How Persistence Changes with ρ



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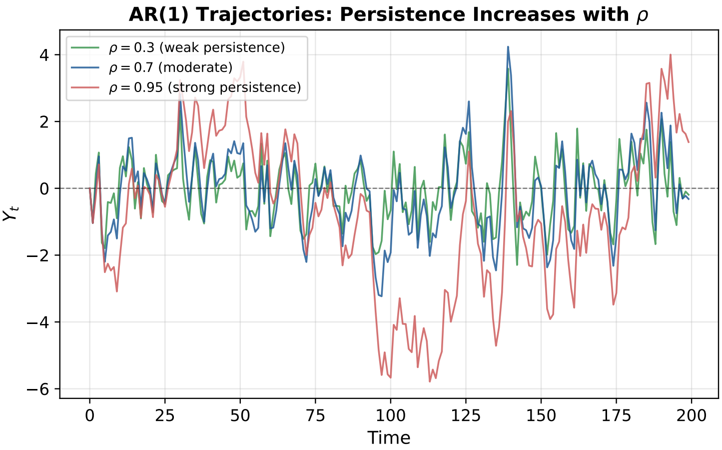

Same shocks, different ρ. Higher $\rho \implies$ slower mean reversion, longer "memory."

Model Selection: AIC and BIC

How many lags? Use **information criteria**:

$$\text{AIC} = \ln(\hat{\sigma}^2) + \frac{2K}{T}, \qquad \text{BIC} = \ln(\hat{\sigma}^2) + \frac{K\ln(T)}{T}$$

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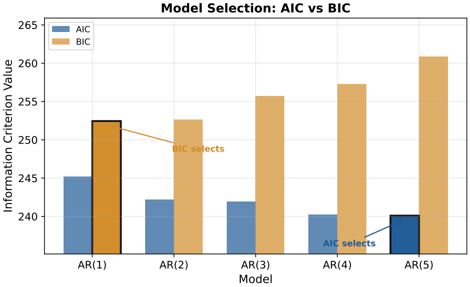

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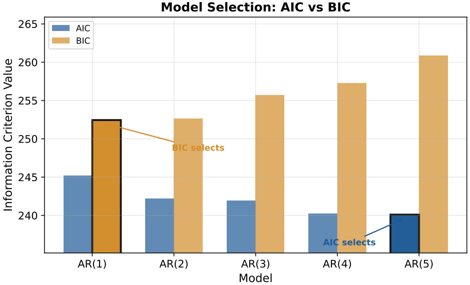

When AIC and BIC disagree, BIC is typically preferred for consistent selection.

Autoregressive Distributed Lag: ARDL(p, q)

An **ARDL** model includes lags of both Y and an explanatory variable X:

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$\Longrightarrow$ Current inflation depends on: (1) its own recent history, and (2) current *and* past unemployment changes.

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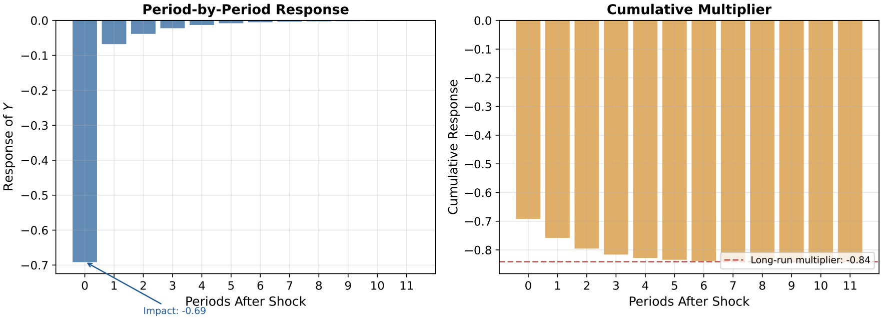
$\Longrightarrow$ The long-run effect is larger (in absolute value) than the impact effect whenever $|\theta_1| > 0$.

Impulse Response: Visualizing the Multipliers

ARDL(1,1) with $\hat{\theta}_1 = 0.56$, $\hat{\delta}_0 = -0.69$, $\hat{\delta}_1 = 0.32$:



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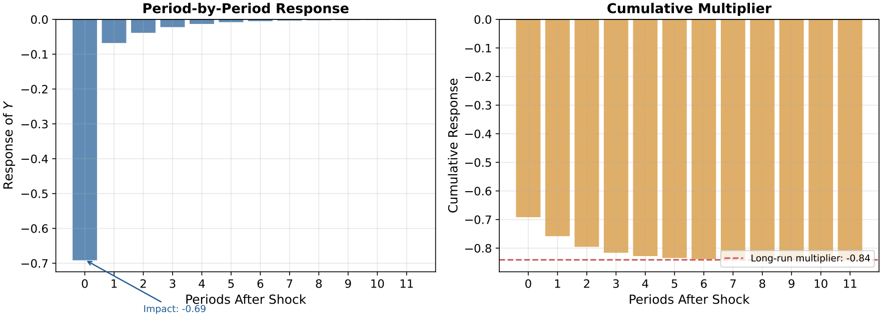

- Impact multiplier: -0.69
- Interim: $-0.69 + 0.32 = -0.37$
- Long-run: $\frac{-0.69+0.32}{1-0.56} = -0.84$

Multiplier Practice Problem

An ARDL(1,1) model for quarterly inflation gives:

$$\widehat{\text{INF}}_t = 1.2 + 0.56\,\text{INF}_{t-1} - 0.69\,\text{DU}_t + 0.32\,\text{DU}_{t-1}$$

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$\implies$ The long-run effect (-0.84) exceeds the impact (-0.69) because the AR term propagates the shock.

Serial Correlation in Regression Residuals

Suppose we estimate:

$$Y_t = \beta_0 + \beta_1 X_t + e_t$$

and the residuals $\hat{e}_t$ are correlated over time: $\text{Cov}(\hat{e}_t, \hat{e}_{t-1}) \neq 0$.

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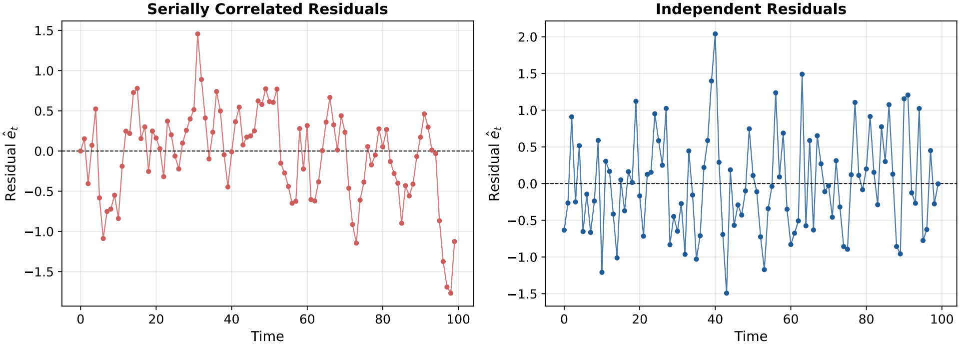
$\implies$ The coefficients are fine, but our **inference** (tests, CIs) is unreliable.

What Serially Correlated Residuals Look Like



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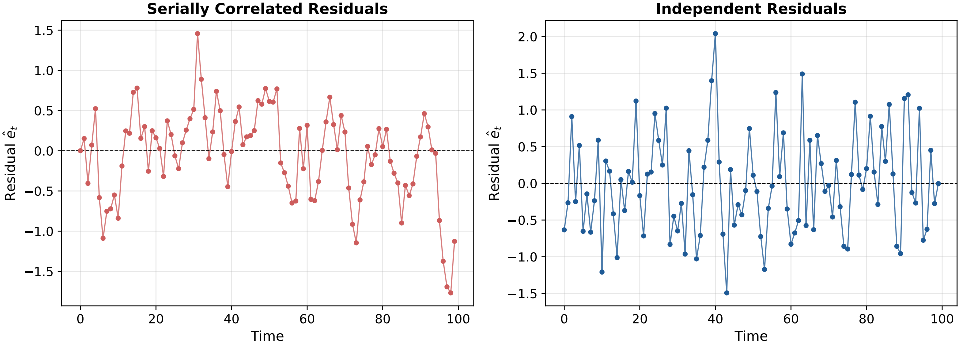

Left: runs of positive then negative residuals $\implies$ positive autocorrelation. **Right:** residuals bounce randomly around zero $\implies$ no serial correlation.

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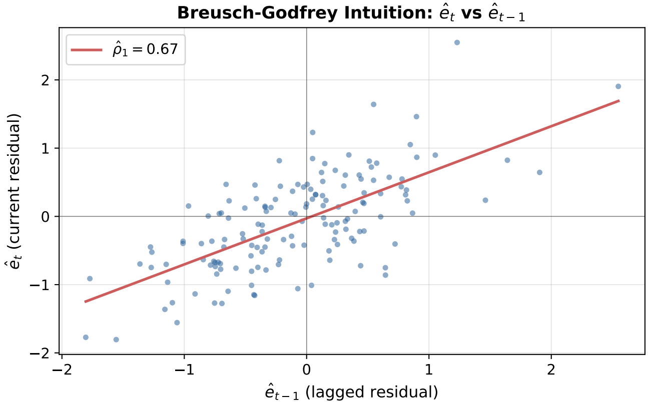
Important: the auxiliary regression *includes all original regressors*. This is what makes BG valid even with lagged dependent variables (unlike Durbin-Watson).

BG Test: The Intuition

If residuals are serially correlated, $\hat{e}_t$ and $\hat{e}_{t-1}$ should be correlated:



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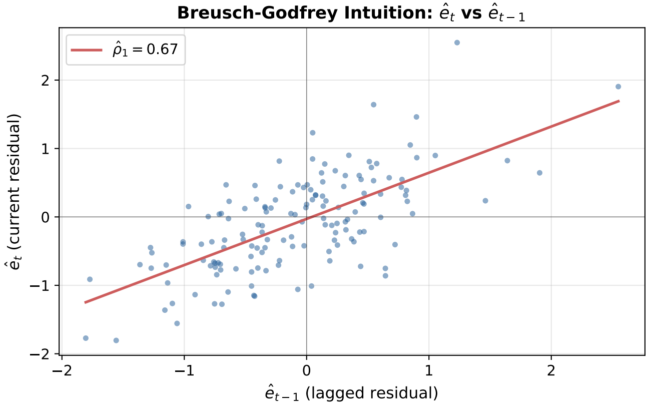

The BG auxiliary regression checks whether $\hat{\rho}_1$ (the slope in this scatter) is significantly different from zero, *after controlling for the original regressors*.

Forecasting with AR Models

One-step-ahead forecast from an AR(2):

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$\Longrightarrow$ Each forecast builds on previous forecasts, compounding uncertainty.

Forecast Practice Problem

An AR(2) model for quarterly inflation gives:

$$\hat{\delta} = 0.4523, \quad \hat{\theta}_1 = 0.6234, \quad \hat{\theta}_2 = 0.2145$$

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Solution:

$$\begin{aligned} \widehat{\text{INF}}_{T+1} &= 0.4523 + 0.6234(2.5) + 0.2145(3.0) \\ &= 0.4523 + 1.5585 + 0.6435 \\ &= 2.65 \end{aligned}$$

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As we forecast further ahead, uncertainty grows:

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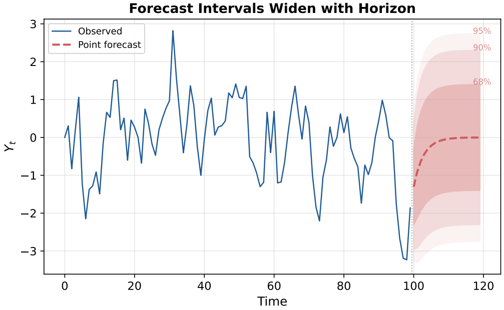

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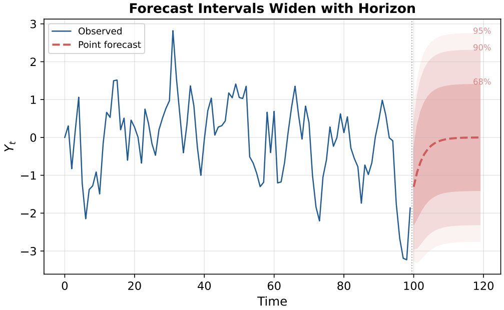

$\implies$ Short-horizon forecasts are relatively precise. Long-horizon forecasts revert toward the unconditional mean with wide bands.

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$\Longrightarrow$ Always evaluate forecasts out-of-sample when the goal is prediction.

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7. **Forecasting:** iterate AR/ARDL forward; confidence bands widen with horizon. Evaluate out-of-sample.

Thank you!

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