Endogenous Regressors

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May 16, 2026

The OLS Assumptions Under Random Sampling

Assumption	**Description**
RS1	The model is linear: $y_i = \beta_1 + \beta_2 x_i + e_i$
RS2	The data (y_i, x_i) are i.i.d.
RS3	**Exogeneity:** $E(e_i \mid x_i) = 0$
RS4	Homoskedasticity: $\text{Var}(e_i \mid x_i) = \sigma^2$
RS5	Rank condition: x_i takes at least two values
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Three sources of endogeneity:

1. Omitted variable bias
2. Measurement error
3. Simultaneity

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Three researchers run different regressions:

1. "The past is irrelevant!" $\implies Y = \beta_0 + \beta_1 X_1 + e_1$
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Question: Will $\hat{\beta}_1$ be the same in Model 1 and Model 3?

A Counterexample: The Room Game

Imagine a game:

- You walk down a hallway with rooms on each side
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$\implies$ OVB requires both (1) omitted variable affects Y, and (2) omitted variable is correlated with included X.

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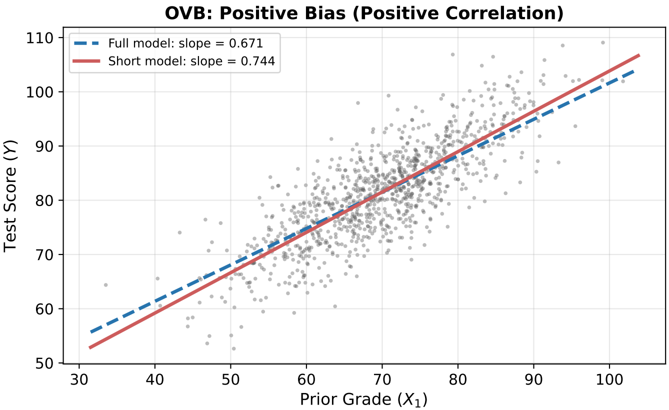
$\implies$ The bias is $\hat{\beta}_2 \times \hat{\delta}_1$. If either is zero, there is no bias.

Visualizing OVB: Positive Bias



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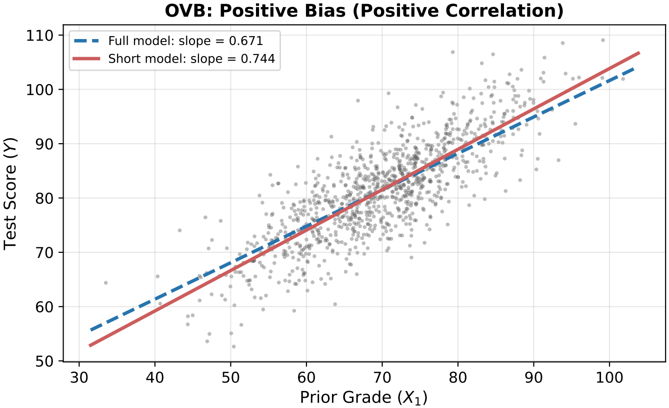
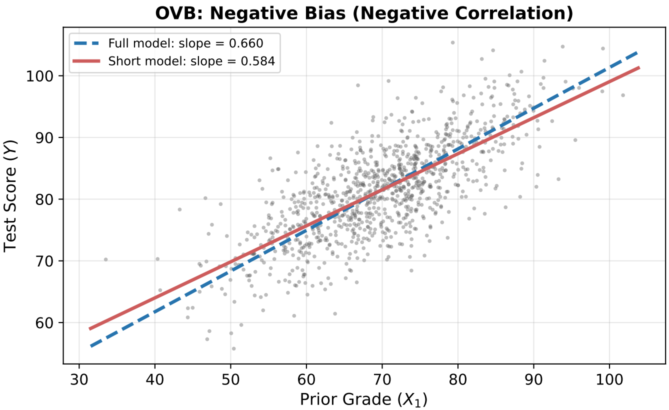


When X_1 and X_2 are **positively** correlated and $\beta_2 > 0$: the short model overestimates β_1 because it gives X_1 credit for the effect of X_2.

Visualizing OVB: Negative Bias



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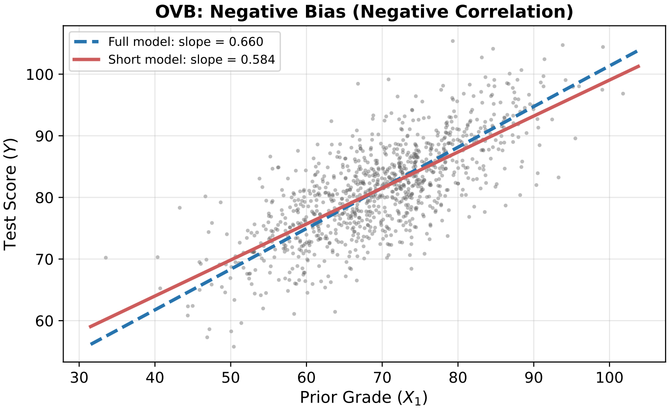

When X_1 and X_2 are **negatively** correlated and $\beta_2 > 0$: the short model underestimates β_1 because the omitted variable works against X_1.

Direction of Bias: The Sign Table

$$\hat{\beta}_1^{\text{short}} = \hat{\beta}_1^{\text{long}} + \hat{\beta}_2 \times \hat{\delta}_1$$

	$\text{Corr}(X_{\textbf{inc}}, X_{\textbf{omit}}) > 0$	$\text{Corr}(X_{\textbf{inc}}, X_{\textbf{omit}}) < 0$
$\beta_{\textbf{omit}} > 0$	**Positive bias (overestimate)**	**Negative bias (underestimate)**
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The direction of OVB depends on two signs: (1) the effect of the omitted variable on Y, and (2) the correlation between included and omitted regressors.

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True DGP: $Y = 30 + 0.65X_1 + 0.40X_2 + \varepsilon$, with $\text{Corr}(X_1, X_2) \approx 0.4$

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$\Longrightarrow$ Both $\beta > 0$ and Corr > 0, so the bias is positive (overestimation) in both short models.

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The proxy measures true study time with error:

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$\implies$ Office hours attendance is a **noisy** version of the true variable we care about.

Why Measurement Error Causes Endogeneity

Substitute $x_i^* = x_i - u_i$ into the true model:

$$y_i = \beta_1 + \beta_2(x_i - u_i) + v_i = \beta_1 + \beta_2 x_i + \underbrace{(v_i - \beta_2 u_i)}_{e_i}$$

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$\Longrightarrow$ If $\beta_2 > 0$, there is a **negative** correlation between x_i and e_i. OLS underestimates β_2.

Attenuation Bias Formula

As $N \to \infty$, the OLS estimator converges to:

$$b_2 \xrightarrow{p} \beta_2 \cdot \underbrace{\frac{\sigma^2_{x^*}}{\sigma^2_{x^*} + \sigma^2_u}}_{\lambda}$$

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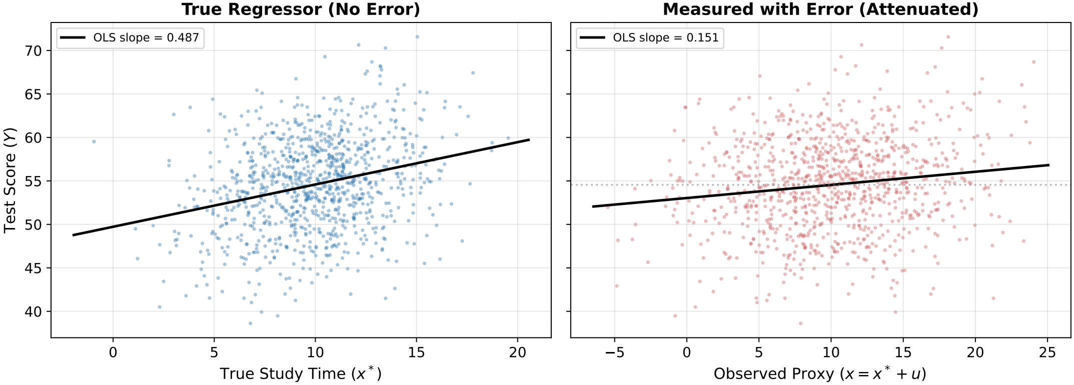
$\implies$ Measurement error **always shrinks the coefficient toward zero**. This is called **attenuation bias**. More data does not help.

Visualizing Measurement Error



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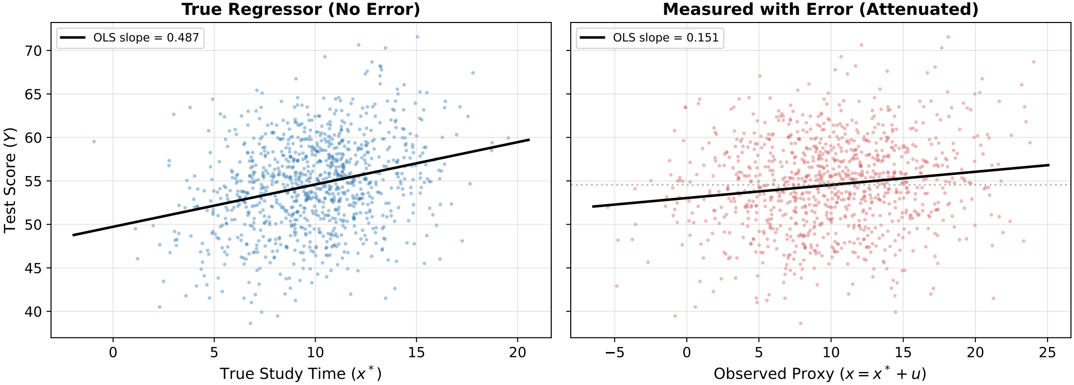
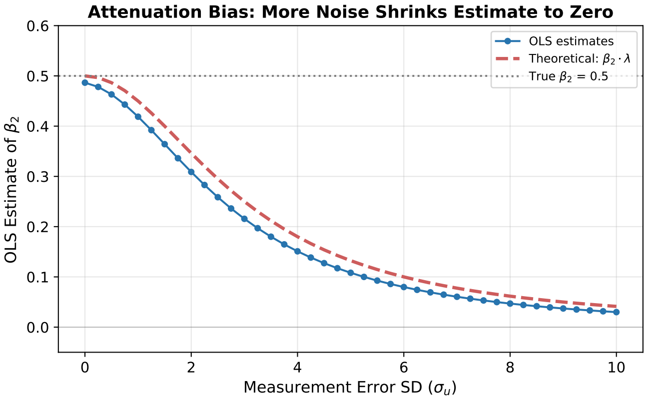


Left: using true x^*, OLS recovers the correct slope. Right: using observed x, the scatter is wider and the slope is attenuated.

Simulation: Attenuation Bias in Action



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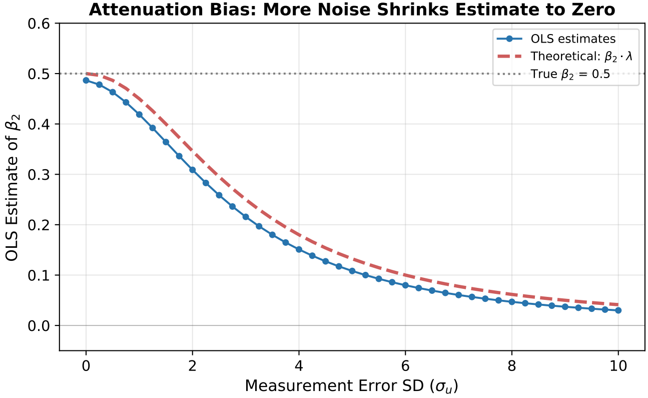
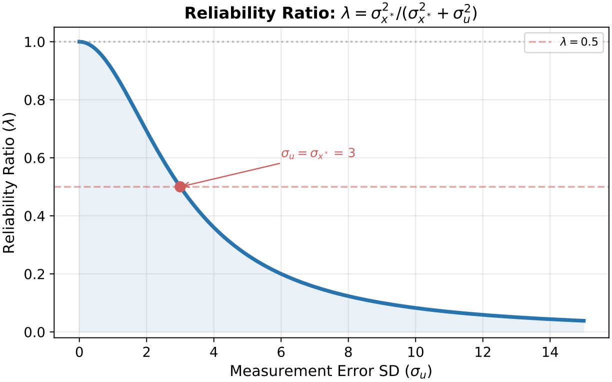


As measurement error (σ_u) increases, the OLS estimate shrinks toward zero. The theoretical curve $\beta_2 \cdot \lambda$ matches the simulated estimates.

The Reliability Ratio



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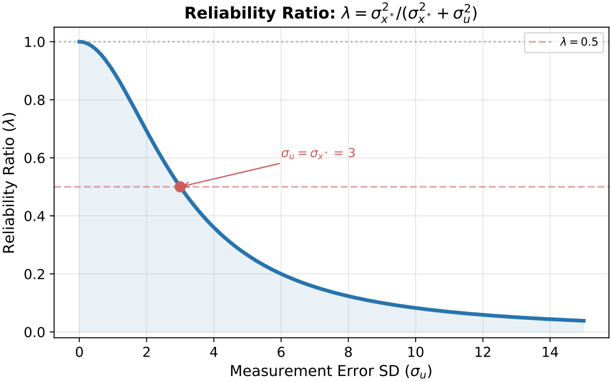
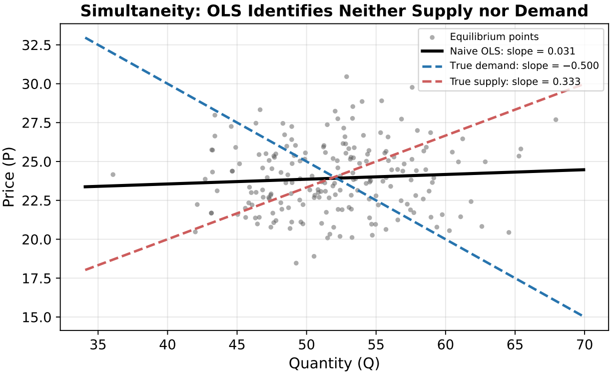


When $\sigma_u = \sigma_{x^*}$, the reliability ratio drops to 0.5: OLS captures only half of the true effect. As measurement error grows, the regressor becomes uninformative.

Simultaneity: A Brief Introduction



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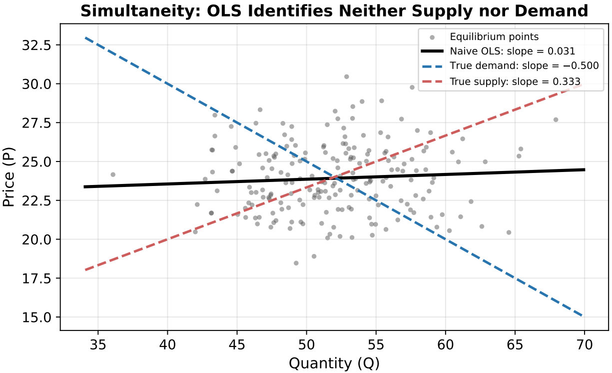

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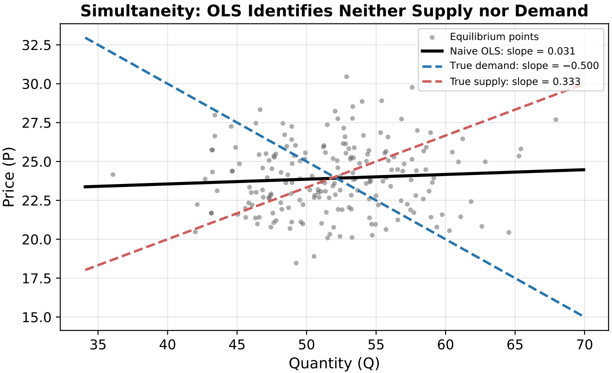

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$\Longrightarrow$ Covered in detail in the **Simultaneous Equations** chapter. Solution: instrumental variables.

Lagged Dependent Variable Models

A common dynamic model:

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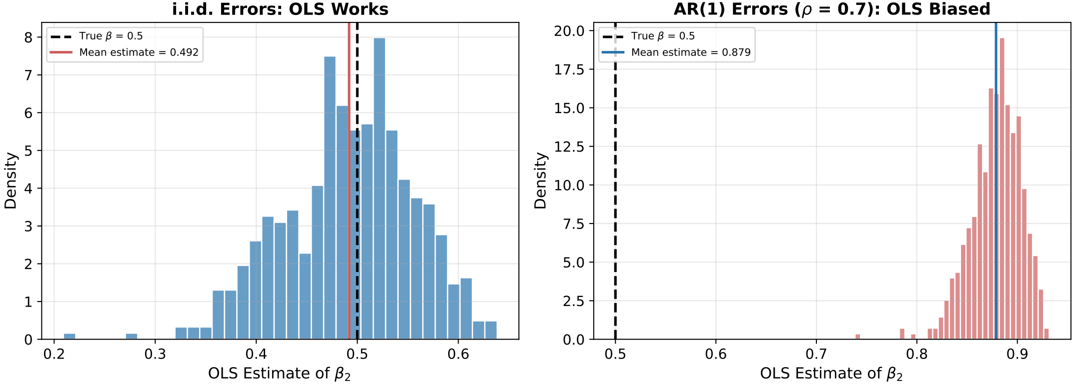
$\implies$ Serial correlation in errors makes y_{t-1} endogenous.

Simulation: Lagged DV with Serial Correlation



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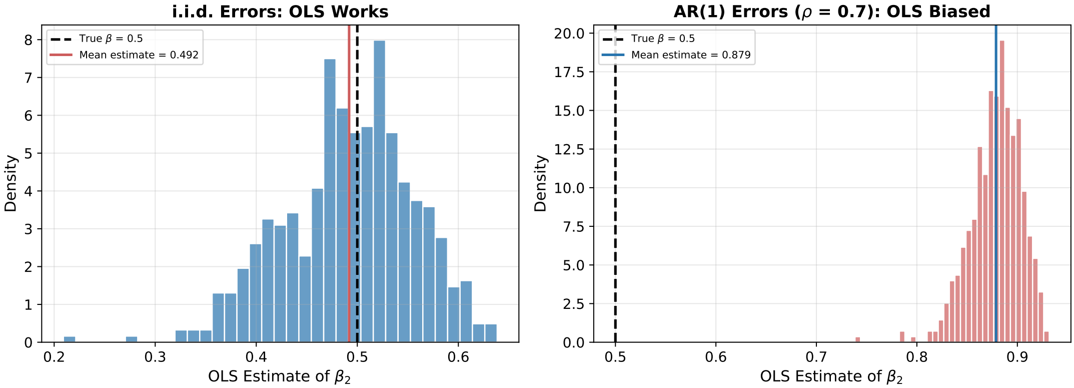

Left: with i.i.d. errors, OLS is centered on the true β. Right: with AR(1) errors ($\rho = 0.7$), OLS is **biased upward**. OLS attributes error persistence to β_2.

Testing for Serial Correlation

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1. Estimate the model and obtain residuals $\hat{e}_t$
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If the test rejects:

- The lagged dependent variable is endogenous
- OLS is inconsistent
- $\implies$ Consider instrumental variables or other methods

Three Sources of Endogeneity

Source	Problem	Bias Direction
Omitted variables	Correlated omitted factor	Depends on signs
Measurement error	Noisy proxy for true X	Toward zero (attenuation)
Simultaneity	X and Y jointly determined	Ambiguous
Lagged DV + AR(1)	y_{t-1} correlated with e_t	Upward (when $\rho > 0$)

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What can we do?

- **Omitted variables**: include the variable, or use instrumental variables / fixed effects
- **Measurement error**: find a better measure, or use IV
- **Simultaneity**: use simultaneous equations methods (2SLS)
- **Lagged DV**: test with Breusch-Godfrey, use IV if errors are autocorrelated

Thank you!

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