Simultaneous Equations

The Chicken and the Egg

Jake Anderson

May 16, 2026

"My belt holds my pants up, but the belt loops hold my belt up.

I don't really know what's happening down there.

Who is the real hero?"

— Mitch Hedberg

Outline

1. Motivation: Why OLS Fails
2. Identification: Tracing Out Curves
3. Structural Form vs Reduced Form
4. The Identification Problem
5. 2SLS for Simultaneous Systems
6. Sargan Overidentification Test
7. Worked Example: Truffle Market

The Problem: Jointly Determined Variables

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- **Quantity sold** also depends on the price
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$\Longrightarrow$ Regressing Q on P with OLS does not recover the demand curve or the supply curve. It produces a confused mixture of both.

Bruin Boba: The Setup

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1	Normal	$4.80	104
2	Hot week (demand $\uparrow$)	$6.00	140
3	Pearl shortage (supply $\downarrow$)	$6.00	80
4	Instagram viral (demand $\uparrow$)	$5.60	128
5	Milk spike (supply $\downarrow$)	$6.40	72

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Notice: Weeks 2 and 3 have the **same price** (\$6.00) but very different quantities (140 vs 80). One is a demand shift, the other is a supply shift.

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$\Longrightarrow$ A single OLS line through (P_t, Q_t) cannot recover demand or supply.

Bruin Boba: The True Curves

The true (linear) curves:

$$\textbf{Demand:} \quad Q = 200 - 20P + u_D$$

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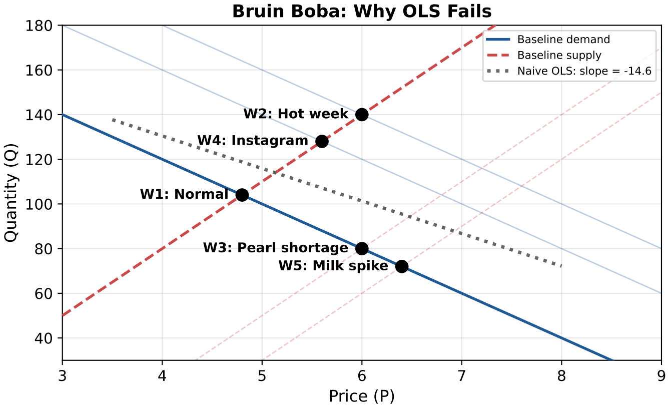
Equilibrium: Set demand = supply:

$$200 - 20P + u_D = -40 + 30P + v_S$$

$$P^* = \frac{240 + u_D - v_S}{50}, \qquad Q^* = 200 - 20P^* + u_D$$

Visualizing the Problem



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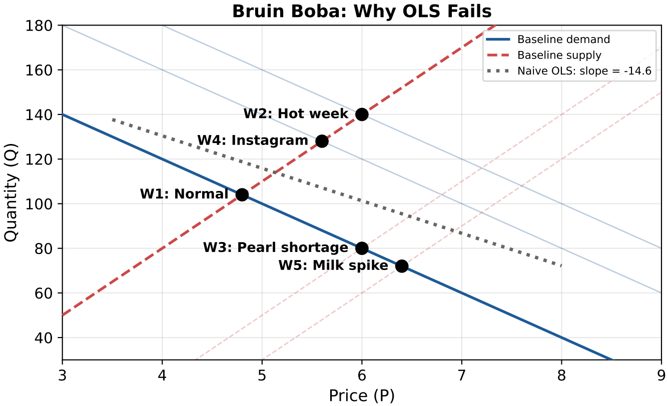

The gray dotted line is naive OLS. It does not match the demand curve (blue) or the supply curve (red). This is **simultaneity bias**.

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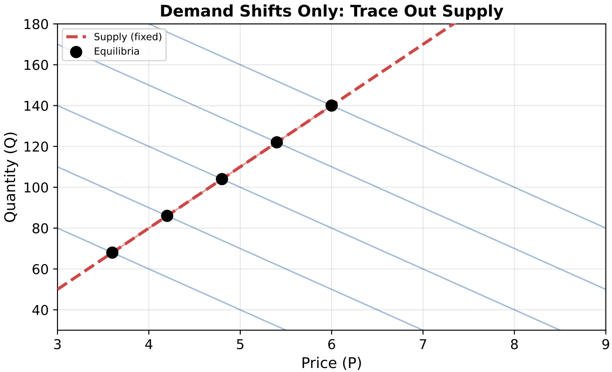

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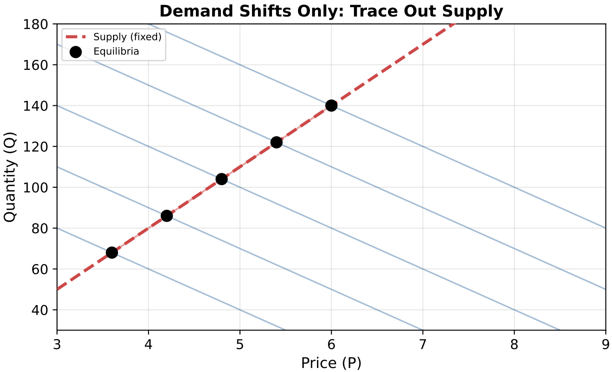

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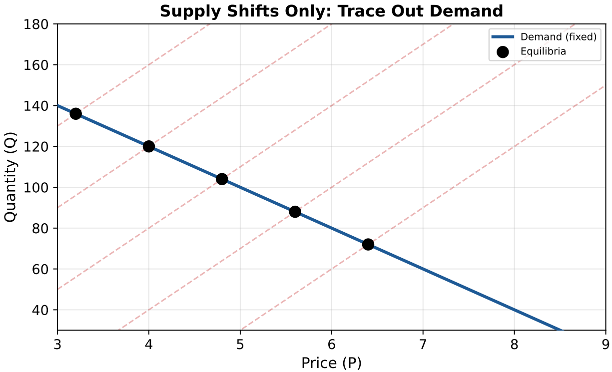

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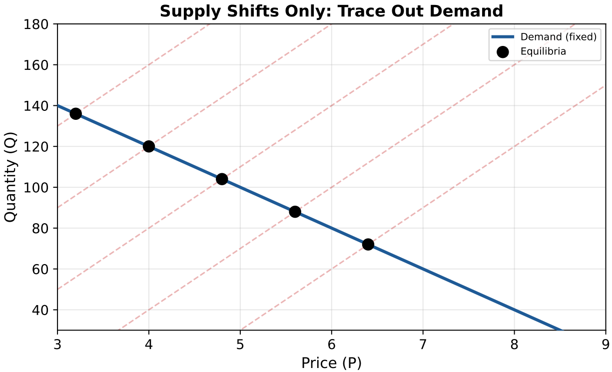

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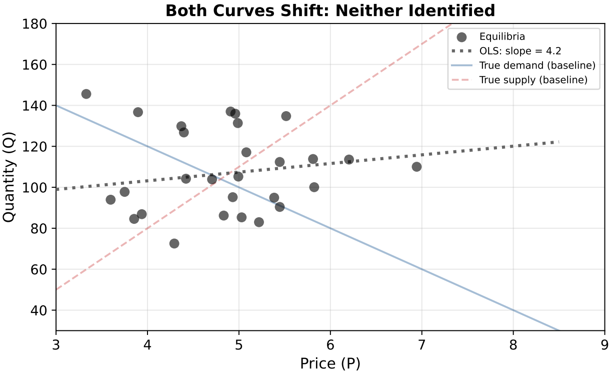

When Both Shift: No Identification

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$\Longrightarrow$ Without excluded instruments, we cannot identify either curve.

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$\Longrightarrow$ In the demand equation, P is correlated with e^d because supply shocks affect P through equilibrium. OLS on structural equations is **biased and inconsistent**.

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Written compactly:

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$\implies$ OLS works on reduced-form equations because the RHS contains only exogenous variables. These are exactly the **first-stage regressions** in 2SLS.

Structural vs Reduced Form: Summary

Structural Form

Demand: $Q = \alpha_1 + \alpha_2 P + \alpha_3 PS + \alpha_4 DI + e^d$

Supply: $Q = \beta_1 + \beta_2 P + \beta_3 PF + e^s$

Solve for equilibrium →

Reduced Form

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The reduced-form π coefficients are combinations of structural α's and β's. Recovering the structural parameters requires identification.

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Exclusions	vs $M-1$	Status	Meaning
$< M-1$	too few	Not identified	Cannot estimate
$= M-1$	exact	Just identified	Exactly enough instruments
$> M-1$	more than enough	Overidentified	Surplus instruments (testable)

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$\implies$ Both equations are identified. The supply equation has a surplus instrument we can test.

Identification: Visual Summary

Identified: Truffle Market

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If both equations contain the same variables, there are no excluded instruments and neither equation is identified.

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Not-identified case. Every observable variable enters *both* equations, so whenever something changes, both curves shift together. The new equilibrium could be anywhere. The scatter of price-quantity points doesn't trace out either curve, just the joint movement. $\Longrightarrow$ No estimator (OLS, 2SLS, anything) can separate demand from supply with this data alone.

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$\Longrightarrow$ The excluded variable is what lets your data answer "which curve is which." Without one, the question is unanswerable, not just hard.

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2SLS: Why Stage 1 Is the Reduced Form for P

We want to estimate the structural demand equation:

$$Q = \alpha_1 + \alpha_2 P + \alpha_3 \mathrm{PS} + \alpha_4 \mathrm{DI} + e^d.$$

OLS is biased because P is endogenous: supply shocks e^s move P, and through the simultaneous system, P ends up correlated with e^d.

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Fix: replace P with a version that depends *only* on the exogenous variables of the system. Get there by setting demand = supply and solving for P:

$$\alpha_1 + \alpha_2 P + \alpha_3 \text{PS} + \alpha_4 \text{DI} + e^d \;=\; \beta_1 + \beta_2 P + \beta_3 \text{PF} + e^s.$$

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$\implies$ This is the **reduced form for** P: P written purely as a function of exogenous variables. Because the right-hand side is exogenous, *OLS on this equation is consistent*. That regression is Stage 1.

2SLS: The Two Stages, Clearly

Stage 1: estimate the reduced form for P and form $\hat{P}$.

$$P = \pi_{10} + \pi_{11}\mathrm{PS} + \pi_{12}\mathrm{DI} + \pi_{13}\mathrm{PF} + v_1 \quad \xrightarrow{\text{OLS}} \quad \hat{P}_i = \hat{\pi}_{10} + \hat{\pi}_{11}\mathrm{PS}_i + \hat{\pi}_{12}\mathrm{DI}_i + \hat{\pi}_{13}\mathrm{PF}_i.$$

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Stage 2: replace P with $\hat{P}$ in the structural demand equation and run OLS.

$$Q = \alpha_1 + \alpha_2\hat{P} + \alpha_3\mathsf{PS} + \alpha_4\mathsf{DI} + \text{residual}.$$

The coefficient on $\hat{P}$ is the 2SLS estimate of α_2 (the demand slope).

2SLS: The Two Stages, Clearly

Stage 1: estimate the reduced form for P and form $\hat{P}$.

$$P = \pi_{10} + \pi_{11}\text{PS} + \pi_{12}\text{DI} + \pi_{13}\text{PF} + v_1 \quad \xrightarrow{\text{OLS}} \quad \hat{P}_i = \hat{\pi}_{10} + \hat{\pi}_{11}\text{PS}_i + \hat{\pi}_{12}\text{DI}_i + \hat{\pi}_{13}\text{PF}_i.$$

$\hat{P}$ is the part of P explained by exogenous variables; the leftover v_1 carries the endogenous noise.

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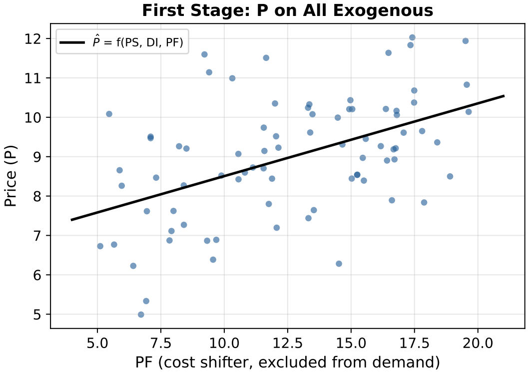
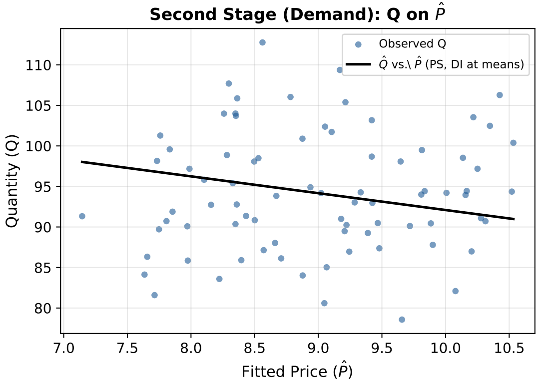
Why it works.

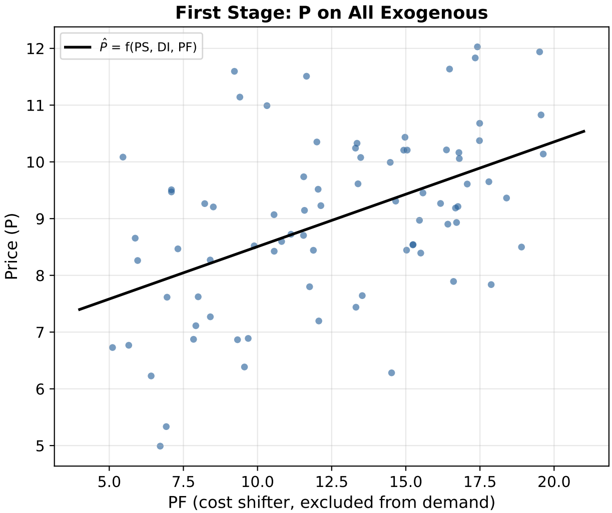
- $\hat{P}$ depends only on exogenous variables, so $\text{Cov}(\hat{P}, e^d) = 0$ in the limit.
- The exclusion that makes 2SLS identify demand is the price of a production input: it appears in $\hat{P}$ (because it shifts supply) but does *not* enter the demand equation itself $\Longrightarrow$ it's the instrument.
- Standard errors should come from a 2SLS-aware routine (e.g. `ivreg` in R), not from running two OLS regressions by hand.

2SLS: The Two Stages Visually





First Stage: What to Notice



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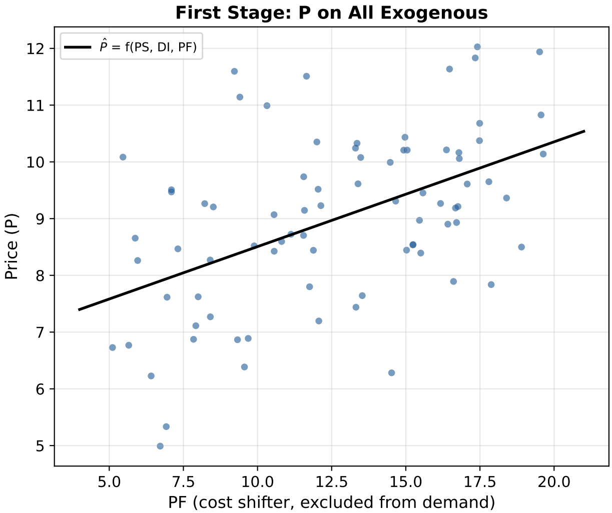

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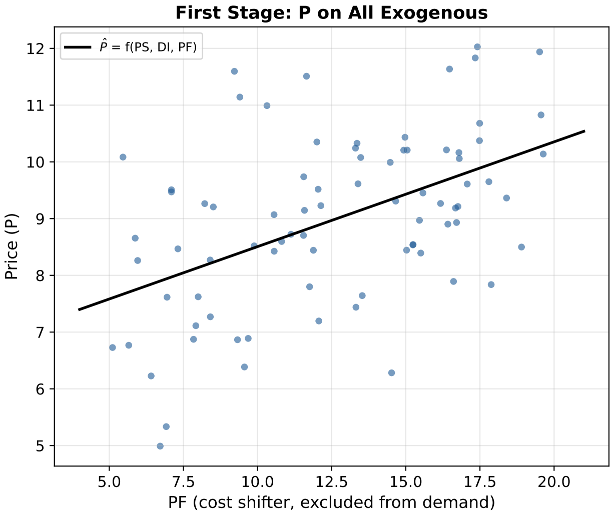
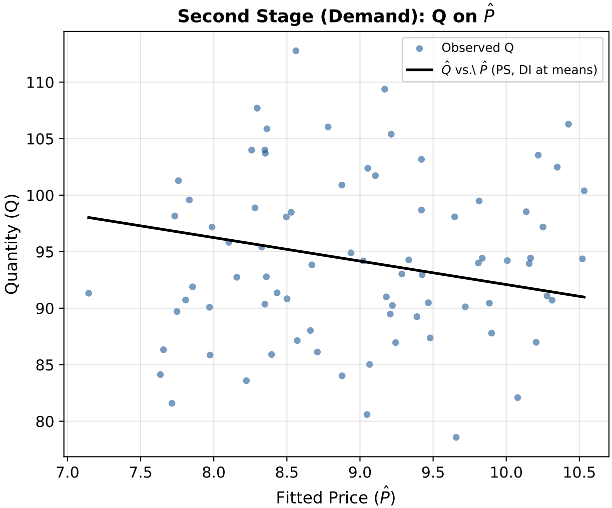


- **Nonzero slope.** The fitted line clearly tilts up $\implies$ the input price moves P. A near-flat line would mean a weak instrument and 2SLS does not work.
- **Sign matches theory.** Higher input cost $\implies$ higher price. The supply curve shifts up, equilibrium price rises.

Second Stage: What to Notice



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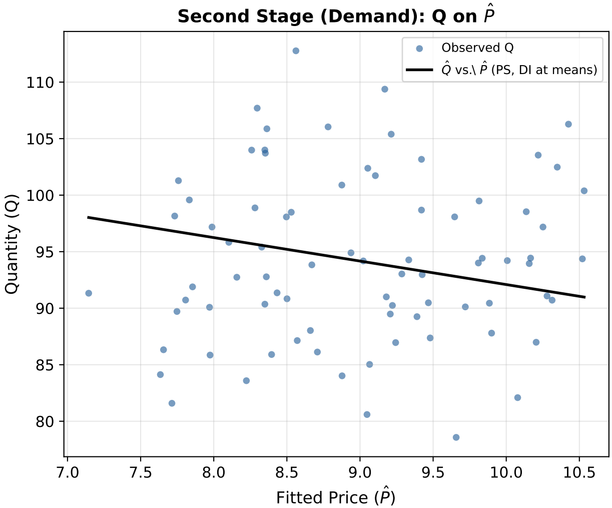

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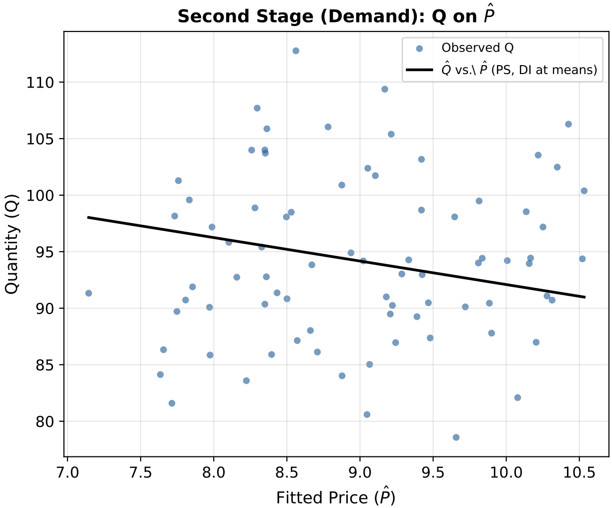

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Same first stage, different second stage. Each structural equation uses different excluded variables as instruments.

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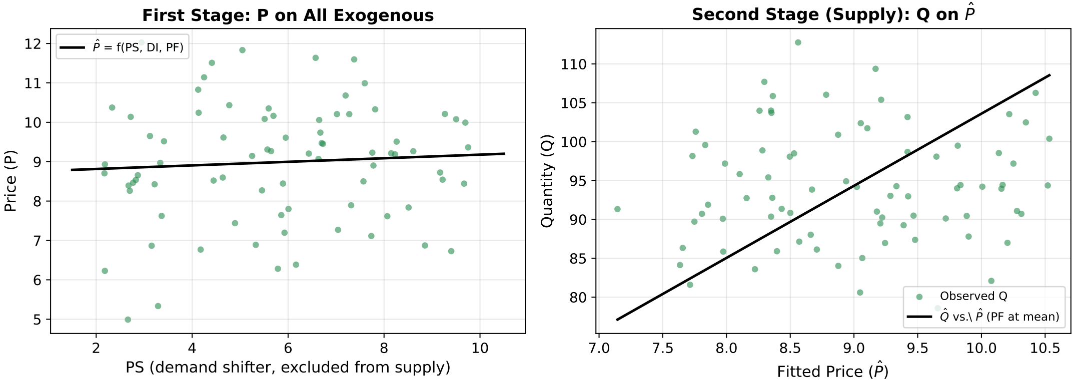

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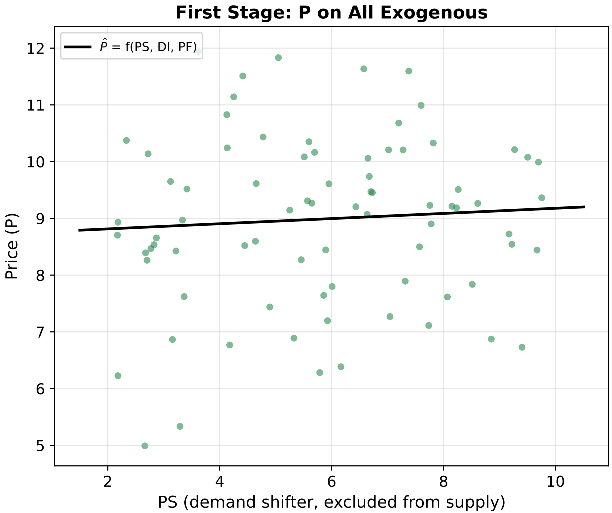

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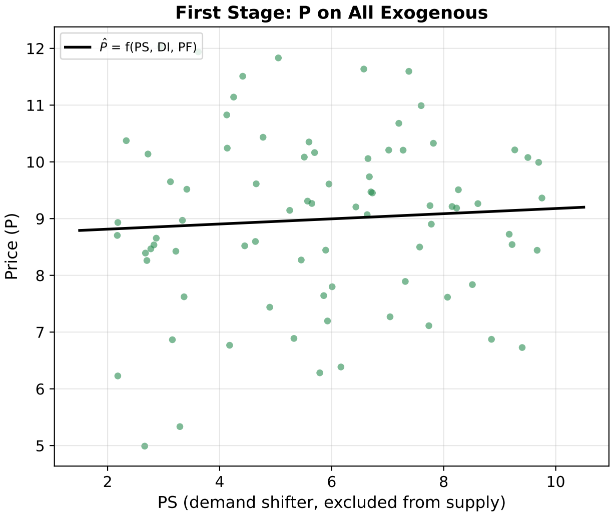

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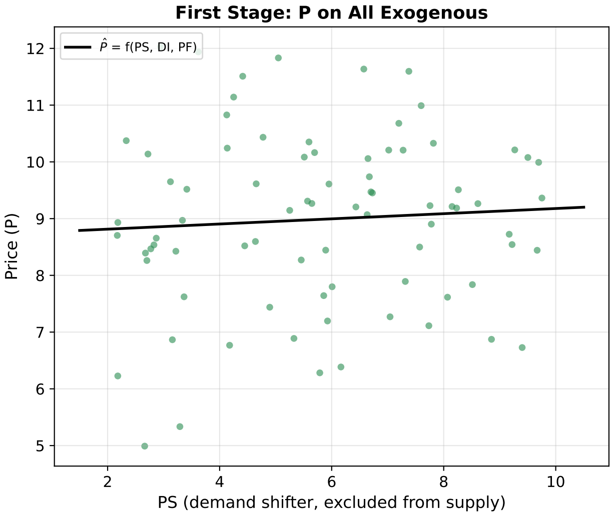

- **Nonzero slope.** PS (the substitute price, excluded from supply) clearly moves $P \implies$ it is a relevant instrument for the supply equation.
- **Sign matches theory.** A higher substitute price shifts demand for our good outward, pushing equilibrium price up.

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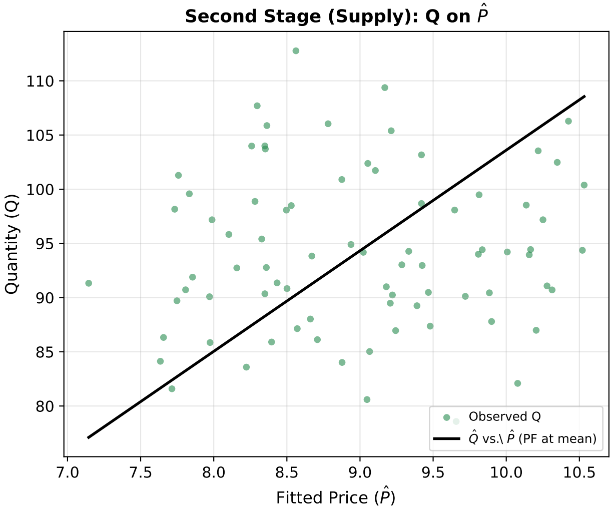

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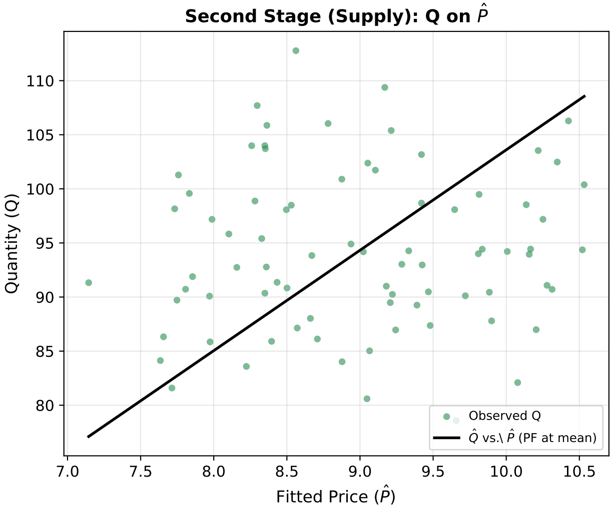

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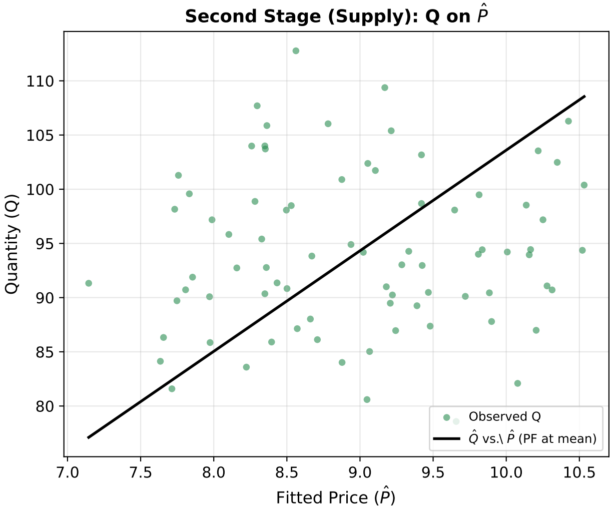

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$\Longrightarrow$ The order condition is *necessary* but not *sufficient*. Instrument strength is equally important.

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4. The Identification Problem
5. 2SLS for Simultaneous Systems
6. Sargan Overidentification Test
7. Worked Example: Truffle Market

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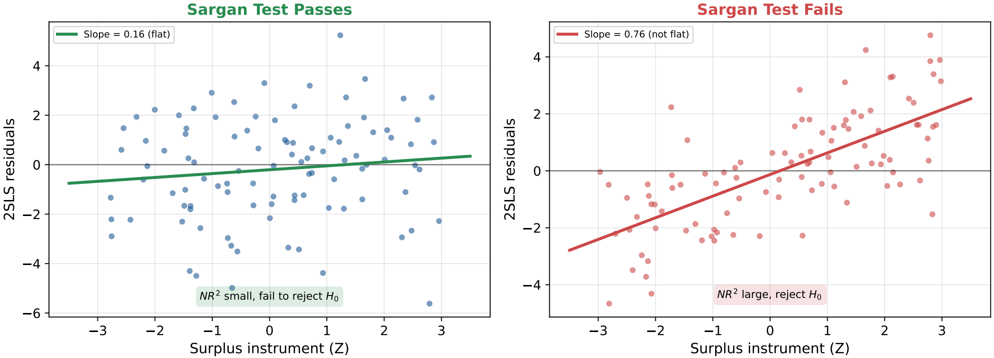
$\Longrightarrow$ If $NR^2 > \chi^2_{L-B,\alpha}$, reject. The surplus instruments appear to be correlated with the error, suggesting misspecification.

Sargan Test: Visual Intuition



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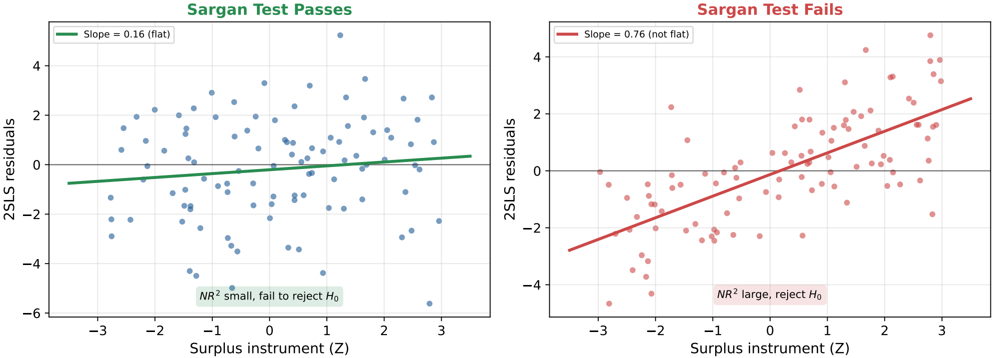

Left: Residuals show no pattern with the surplus instrument $\implies$ valid. **Right:** Clear correlation $\implies$ instrument is endogenous.

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Limitation: The Sargan test assumes at least one instrument is valid. If *all* instruments are bad, the test has no power to detect it.

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Structural model:

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- Demand: excludes PF (1 exclusion $\geq M - 1 = 1$) $\implies$ just identified
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$\Longrightarrow$ Trust the 2SLS estimate (-0.53). Since PF shifts supply but not demand, it isolates movement along the demand curve.

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$\Longrightarrow$ The instruments appear valid. No evidence of misspecification in the supply equation.

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Thank you!

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